

Exceptional Lie groups

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Preface

In the end of 19 century, W. Killing and E. Cartan classified the complex simple Lie algebras, called A_n, B_n, C_n, D_n (classical type) and G_2, F_4, E_6, E_7, E_8 (exceptional type). These simple Lie algebras and the corresponding compact simple Lie groups have offered many subjects in mathematicians. Especially, exceptional Lie groups are very wonderful and interesting miracle in Lie group theory.

Now, in the present book, we describe simply connected compact exceptional simple Lie groups G_2, F_4, E_6, E_7, E_8 , in very elementary way. The contents are given as follows. We first construct all simply connected compact exceptional Lie groups G concretely. Next, we find all involutive automorphisms σ of G , and determine the group structures of the fixed points subgroup G^σ by σ . Note that they correspond to classification of all irreducible compact symmetric spaces G/G^σ of exceptional type, and that they also correspond to classification of all non-compact exceptional simple Lie groups. Finally, we determined the group structures of the maximal subgroups of maximal rank. At any rate, we would like this book to be used in mathematics and physics.

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Notation

$\mathbf{R}, \mathbf{C} = \mathbf{R} \oplus \mathbf{R}e_1, \mathbf{H} = \mathbf{R} \oplus \mathbf{R}e_1 \oplus \mathbf{R}e_2 \oplus \mathbf{R}e_3$ denote the fields of real, complex, quaternion numbers, respectively.

For \mathbf{R} -vector space V , its complexification $\{u + iv \mid u, v \in V\}$ is denoted by $V^{\mathbf{C}}$. The complex conjugation in $V^{\mathbf{C}}$ is denoted by τ :

$$\tau(u + iv) = u - iv.$$

$\mathbf{R}^{\mathbf{C}}$ is briefly denoted by \mathbf{C} .

For K -vector space V ($K = \mathbf{R}, \mathbf{C}, \mathbf{C}$), $\text{Iso}_K(V)$ denotes all K -linear isomorphisms of V . For a K -linear mapping f of V , V_f denotes $\{v \in V \mid f(v) = v\}$.

For K -vector spaces V, W ($K = \mathbf{R}, \mathbf{C}$), $\text{Hom}_K(V, W)$ denotes all K -homomorphism $f : V \rightarrow W$. $\text{Hom}_K(V, V)$ is briefly denoted by $\text{Hom}_K(V)$.

Let G be a group and σ an automorphism of G . Then G^σ denotes $\{g \in G \mid \sigma(g) = g\}$. For $s \in G$, G^s denotes $\{g \in G \mid sgs^{-1} = g\}$.

For topological spaces X, Y , $X \simeq Y$ denotes that X and Y are homeomorphic.

For groups G, G' , $G \cong G'$ denotes that G and G' are isomorphic as groups. Isomorphic two groups G, G' are often identified: $G = G'$.

$M(n, K)$ denotes all $n \times n$ matrices with entries in K .

$E = \text{diag}(1, \dots, 1) \in M(n, K)$ is the unit matrix.

For $A \in M(n, K)$, ${}^t A$ denotes the transposed matrix of A and A^* denotes the conjugate transposed matrix of A : $A^* = {}^t \overline{A}$.

$O(n) = \{A \in M(n, \mathbf{R}) \mid {}^t AA = E\}$ (orthogonal group),

$SO(n) = \{A \in O(n) \mid \det A = 1\}$ (special orthogonal group),

$U(n) = \{A \in M(n, \mathbf{C}) \mid A^* A = E\}$ or $\{A \in M(n, \mathbf{C}) \mid \tau({}^t A)A = E\}$ (unitary group),

$SU(n) = \{A \in U(n) \mid \det A = 1\}$ (special unitary group),

$Sp(n) = \{A \in M(n, \mathbf{H}) \mid A^* A = E\}$ (symplectic group).

For a Lie group G , its Lie algebra is denoted by the corresponding small German letter \mathfrak{g} . For example, $\mathfrak{su}(n)$ is the Lie algebra of the special unitary group $SU(n)$.

Special notations

$$\gamma, \quad \sigma, \quad \iota, \quad \kappa, \quad \lambda, \quad \mu, \quad \tau, \quad \nu, \quad \chi, \quad g, \quad w.$$

Exceptional Lie group G_2

1.1. Cayley algebra \mathfrak{C}

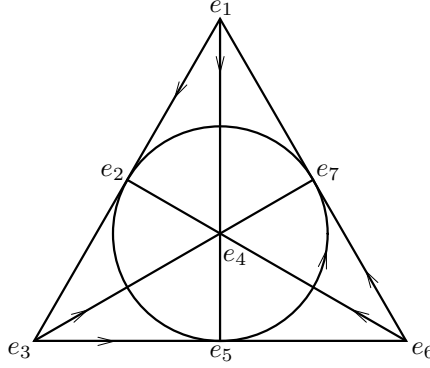
We denote the division Cayley algebra by \mathfrak{C} . We now explain this algebra. Consider an 8 dimensional \mathbf{R} -vector space with basis $\{e_0 = 1, e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$ and define a multiplication between them as follows. In the figure below, the multiplication between e_1, e_2, e_3 is defined as

$$e_1 e_2 = e_3, \quad e_2 e_3 = e_1, \quad e_3 e_1 = e_2,$$

and defined similarly on the other lines. For example, $e_1 e_6 = e_7, e_4 e_7 = e_3$ etc. We regard that e_2, e_5, e_7 are also collinear, for example, $e_5 e_7 = e_2$. $e_0 = 1$ is the unit of the multiplication and assume

$$e_i^2 = -1, \quad i \neq 0, \quad e_i e_j = -e_j e_i, \quad i \neq j, i \neq 0, j \neq 0,$$

and the distributive law. Thus \mathfrak{C} has a multiplication. $x1, x \in \mathbf{R}$ is briefly denoted by x . In \mathfrak{C} , the conjugate \bar{x} , an inner product (x, y) , the length $|x|$ and the real part $R(x)$ are defined respectively by



$$\overline{x_0 + \sum_{i=1}^7 x_i e_i} = x_0 - \sum_{i=1}^7 x_i e_i, \quad \left(\sum_{i=0}^7 x_i e_i, \sum_{i=0}^7 y_i e_i \right) = \sum_{i=0}^7 x_i y_i,$$

$$|x| = \sqrt{(x, x)}, \quad R\left(x_0 + \sum_{i=1}^7 x_i e_i\right) = x_0.$$

For $x \in \mathfrak{C}$, $x \neq 0$, we denote $\frac{\bar{x}}{|x|^2}$ by x^{-1} , then we have $xx^{-1} = x^{-1}x = 1$, and \mathfrak{C} satisfies all axioms of a field except the associative law $x(yz) = (xy)z$. Of course the commutative law $xy = yx$ does not hold. Since the associative law does not

hold, calculations in \mathfrak{C} are complicated, however the following relations hold. (See Freudenthal [6] or Yokota [58]). For $a, b, x, y \in \mathfrak{C}$, we have

- 1 $(xy, xy) = (x, x)(y, y), \quad |xy| = |x||y|.$
- 2 $(ax, ay) = (a, a)(x, y) = (xa, ya).$
- 3 $(ax, by) + (bx, ay) = 2(a, b)(x, y),$
- 4 $(ax, y) = (x, \bar{a}y), \quad (xa, y) = (x, y\bar{a}).$
- 5 $\bar{\bar{x}} = x, \quad \overline{x+y} = \bar{x} + \bar{y}, \quad \overline{xy} = \bar{y}\bar{x}.$
- 6 $(x, y) = (y, x) = \frac{1}{2}(\bar{x}y + \bar{y}x) = \frac{1}{2}(x\bar{y} + y\bar{x}), \quad \bar{x}x = x\bar{x} = |x|^2.$
- 7 $a(\bar{a}x) = (a\bar{a})x, \quad a(x\bar{a}) = (ax)\bar{a}, \quad x(a\bar{a}) = (xa)\bar{a}.$
 $a(ax) = (aa)x, \quad a(xa) = (ax)a, \quad x(aa) = (xa)a,$
- 8 $\bar{b}(ax) + \bar{a}(bx) = 2(a, b)x = (xa)\bar{b} + (xb)\bar{a}.$
- 9 We use a notation $\{x, y, z\} = (xy)z - x(yz)$, called the associator of x, y, z .

Then, we have

$$\{x, y, a\} = \{y, a, x\} = \{a, x, y\} = -\{y, x, a\} = -\{x, a, y\} = -\{a, y, x\}.$$

For example, we have

$$\begin{aligned} (ax)y + x(ya) &= a(xy) + (xy)a, \\ (xa)y + (xy)a &= x(ay) + x(ya), \\ (ax)y + (xa)y &= a(xy) + x(ay). \end{aligned}$$

- 10 $(ax)(ya) = a(xy)a$ (Moufang's formula).
- 11 $R(xy) = R(yx), \quad R(x(yz)) = R(y(zx)) = R(z(xy)) (= R(xyz)).$

For an orthonormal basis $\{1, a_1, a_2, \dots, a_7\}$ of \mathfrak{C} with respect to the inner product (x, y) , the following 12.1 ~ 12.3 hold.

- 12.1 $a_i(a_jx) = -a_j(a_ix), \quad \text{in particular, } a_ia_j = -a_ja_i, \quad i \neq j.$
- 12.2 $a_i(a_ix) = -x, \quad \text{in particular, } a_i^2 = -1.$
- 12.3 $a_i(a_ja_k) = a_j(a_ka_i) = a_k(a_ia_j), \quad i, j, k \text{ are distinct.}$

1.2. Compact exceptional Lie group G_2

Definition. The group G_2 is defined to be the automorphism group of the Cayley algebra \mathfrak{C} :

$$G_2 = \{\alpha \in \text{Iso}_{\mathbf{R}}(\mathfrak{C}) \mid \alpha(xy) = (\alpha x)(\alpha y)\}.$$

Lemma 1.2.1. For $\alpha \in G_2$, we have

$$(\alpha x, \alpha y) = (x, y), \quad x, y \in \mathfrak{C}.$$

Proof. For $\alpha \in G_2$ and $x \in \mathfrak{C}$, we have

$$\overline{\alpha x} = \alpha \bar{x}.$$

To prove this, it is sufficient to show

$$\alpha 1 = 1, \quad \overline{\alpha e_i} = -\alpha e_i, \quad i = 1, 2, \dots, 7.$$

From $(\alpha 1)(\alpha 1) = \alpha(1 \cdot 1) = \alpha 1$ and $\alpha 1 \neq 0$, we have $\alpha 1 = 1$, while from $(\alpha e_i)(\alpha e_i) = \alpha(e_i e_i) = \alpha(-1) = -\alpha 1 = -1$ for $i \neq 0$, we get $\overline{\alpha e_i} = -\alpha e_i$. Now,

$$\begin{aligned} (\alpha x, \alpha y) &= \frac{1}{2}((\alpha x)(\overline{\alpha y}) + (\alpha y)(\overline{\alpha x})) = \frac{1}{2}((\alpha x)(\alpha \overline{y}) + (\alpha y)(\alpha \overline{x})) \\ &= \alpha \left(\frac{1}{2}(x \overline{y} + y \overline{x}) \right) = \alpha((x, y)) = (x, y). \end{aligned}$$

Theorem 1.2.2. G_2 is a compact Lie group.

Proof. G_2 is a compact Lie group as a closed subgroup of the orthogonal group

$$O(8) = O(\mathfrak{C}) = \{\alpha \in \text{Iso}_{\mathbf{R}}(\mathfrak{C}) \mid (\alpha x, \alpha y) = (x, y)\}.$$

Remark. Since $\alpha 1 = 1$ for $\alpha \in G_2$, G_2 is a subgroup of the orthogonal group $O(7) = \{\alpha \in O(\mathfrak{C}) \mid \alpha 1 = 1\}$, that is, $G_2 \subset O(7)$.

1.3. Outer automorphisms of Lie algebra \mathfrak{D}_4

In order to study the Lie algebra \mathfrak{g}_2 of the group G_2 , we consider the Lie algebra

$$\mathfrak{D}_4 = \mathfrak{so}(8) = \mathfrak{so}(\mathfrak{C}) = \{D \in \text{Hom}_{\mathbf{R}}(\mathfrak{C}) \mid (Dx, y) + (x, Dy) = 0\}$$

of the Lie group $SO(8)$.

We define \mathbf{R} -linear mappings $G_{ij} : \mathfrak{C} \rightarrow \mathfrak{C}$, $i, j = 0, 1, \dots, 7, i \neq j$ satisfying

$$G_{ij}e_j = e_i, \quad G_{ij}e_i = -e_j, \quad G_{ij}e_k = 0, \quad k \neq i, j.$$

Then $G_{ij} \in \mathfrak{D}_4$ and $\{G_{ij} \mid 0 \leq i < j \leq 7\}$ forms an \mathbf{R} -basis of \mathfrak{D}_4 . Furthermore we define \mathbf{R} -linear mappings $F_{ij} : \mathfrak{C} \rightarrow \mathfrak{C}$, $i, j = 0, 1, \dots, 7, i \neq j$ by

$$F_{ij}x = \frac{1}{2}e_i(\overline{e_j}x), \quad x \in \mathfrak{C}.$$

Lemma 1.3.1. For $i, j = 0, 1, \dots, 7, i \neq j$, we have $F_{ij} \in \mathfrak{D}_4$, and when $i < j$, F_{ij} is expressed in terms of G_{ij} as follows.

$$\begin{cases} 2F_{01} = G_{01} + G_{23} + G_{45} + G_{67} \\ 2F_{23} = G_{01} + G_{23} - G_{45} - G_{67} \\ 2F_{45} = G_{01} - G_{23} + G_{45} - G_{67} \\ 2F_{67} = G_{01} - G_{23} - G_{45} + G_{67}, \end{cases} \quad \begin{cases} 2F_{02} = G_{02} - G_{13} - G_{46} + G_{57} \\ 2F_{13} = -G_{02} + G_{13} - G_{46} + G_{57} \\ 2F_{46} = -G_{02} - G_{13} + G_{46} + G_{57} \\ 2F_{57} = G_{02} + G_{13} + G_{46} + G_{57}, \end{cases}$$

$$\begin{cases} 2F_{03} = G_{03} + G_{12} + G_{47} + G_{56} \\ 2F_{12} = G_{03} + G_{12} - G_{47} - G_{56} \\ 2F_{47} = G_{03} - G_{12} + G_{47} - G_{56} \\ 2F_{56} = G_{03} - G_{12} - G_{47} + G_{56}, \end{cases} \quad \begin{cases} 2F_{04} = G_{04} - G_{15} + G_{26} - G_{37} \\ 2F_{15} = -G_{04} + G_{15} + G_{26} - G_{37} \\ 2F_{26} = G_{04} + G_{15} + G_{26} + G_{37} \\ 2F_{37} = -G_{04} - G_{15} + G_{26} + G_{37}, \end{cases} \\
\begin{cases} 2F_{05} = G_{05} + G_{14} - G_{27} - G_{36} \\ 2F_{14} = G_{05} + G_{14} + G_{27} + G_{36} \\ 2F_{27} = -G_{05} + G_{14} + G_{27} - G_{36} \\ 2F_{36} = -G_{05} + G_{14} - G_{27} + G_{36}, \end{cases} \quad \begin{cases} 2F_{06} = G_{06} - G_{17} - G_{24} + G_{35} \\ 2F_{17} = -G_{06} + G_{17} - G_{24} + G_{35} \\ 2F_{24} = -G_{06} - G_{17} + G_{24} + G_{35} \\ 2F_{35} = G_{06} + G_{17} + G_{24} + G_{35}, \end{cases} \\
\begin{cases} 2F_{07} = G_{07} + G_{16} + G_{25} + G_{34} \\ 2F_{16} = G_{07} + G_{16} - G_{25} - G_{34} \\ 2F_{25} = G_{07} - G_{16} + G_{25} - G_{34} \\ 2F_{34} = G_{07} - G_{16} - G_{25} + G_{34}. \end{cases}$$

In particular, $\{F_{ij} \mid 0 \leq i < j \leq 7\}$ forms an \mathbf{R} -basis of \mathfrak{D}_4 .

Definition. We define \mathbf{R} -linear mappings $\kappa, \pi, \nu : \mathfrak{D}_4 \rightarrow \mathfrak{D}_4$ respectively by

$$\begin{aligned}
(\kappa D)x &= \overline{Dx}, \quad x \in \mathfrak{C}, \\
\pi(G_{ij}) &= F_{ij}, \quad i, j = 0, 1, \dots, 7, i \neq j, \\
\nu &= \pi\kappa.
\end{aligned}$$

Lemma 1.3.2. The mappings κ, π, ν are automorphisms of the Lie algebra \mathfrak{D}_4 :

$$\kappa, \pi, \nu \in \text{Aut}(\mathfrak{D}_4).$$

Proof. Since $\kappa^2 = 1$, κ is an \mathbf{R} -linear isomorphism of \mathfrak{D}_4 . We have

$$\begin{aligned}
[\kappa D_1, \kappa D_2]x &= (\kappa D_1)(\kappa D_2 x) - (\kappa D_2)(\kappa D_1 x) = \overline{D_1(\overline{\kappa D_2 x})} - \overline{D_2(\overline{\kappa D_1 x})} \\
&= \overline{D_1 D_2 x} - \overline{D_2 D_1 x} = \kappa(D_1 D_2 - D_2 D_1)x = \kappa[D_1, D_2]x, \quad x \in \mathfrak{C}.
\end{aligned}$$

Hence $\kappa \in \text{Aut}(\mathfrak{D}_4)$. Since π maps the \mathbf{R} -basis $\{G_{ij} \mid 0 \leq i < j \leq 7\}$ to the \mathbf{R} -basis $\{F_{ij} \mid 0 \leq i < j \leq 7\}$ of \mathfrak{D}_4 , π is an \mathbf{R} -linear isomorphism. To show that π is an automorphism of \mathfrak{D}_4 , it is sufficient to show that

$$[\pi G_{ij}, \pi G_{kl}] = \pi[G_{ij}, G_{kl}],$$

which in turn would follow from the relations

$$\begin{aligned}
[F_{ij}, F_{jk}] &= F_{ik}, \quad i, j, k \text{ are distinct}, \\
[F_{ij}, F_{kl}] &= 0, \quad i, j, k, l \text{ are distinct}.
\end{aligned}$$

In the following calculations, let i, j, k, l be all distinct and $i, j, k, l \neq 0$. For $x \in \mathfrak{C}$,

$$\begin{aligned}
[F_{i0}, F_{0k}]x &= (F_{i0}F_{0k} - F_{0k}F_{i0})x = -\frac{1}{4}e_i(e_k x) + \frac{1}{4}e_k(e_i x) \\
&= \frac{1}{2}e_k(e_i x) = F_{ik}x,
\end{aligned}$$

$$\begin{aligned}
[F_{i0}, F_{kl}]x &= (F_{i0}F_{kl} - F_{kl}F_{i0})x = \frac{1}{4}e_i(e_l(e_kx)) - \frac{1}{4}e_l(e_k(e_ix)) \\
&= -\frac{1}{4}e_l(e_i(e_kx)) + \frac{1}{4}e_l(e_i(e_kx)) = 0, \\
[F_{ij}, F_{jk}]x &= (F_{ij}F_{jk} - F_{jk}F_{ij})x = \frac{1}{4}e_j(e_i(e_k(e_jx))) - \frac{1}{4}e_k(e_j(e_j(e_ix))) \\
&= \frac{1}{4}e_i(e_k(e_j(e_jx))) + \frac{1}{4}e_k(e_ix) = -\frac{1}{4}e_i(e_kx) + \frac{1}{4}e_k(e_ix) \\
&= \frac{1}{2}e_k(e_ix) = F_{ik}x, \\
[F_{ij}, F_{kl}]x &= (F_{ij}F_{kl} - F_{kl}F_{ij})x = \frac{1}{4}e_j(e_i(e_l(e_kx))) - \frac{1}{4}e_l(e_k(e_j(e_ix))) \\
&= \frac{1}{4}e_j(e_k(e_i(e_lx))) - \frac{1}{4}e_j(e_l(e_k(e_ix))) = \dots = 0.
\end{aligned}$$

Hence $\pi \in \text{Aut}(\mathfrak{D}_4)$. Finally, since $\nu = \pi\kappa$, we have $\nu \in \text{Aut}(\mathfrak{D}_4)$.

Definition. For $a \in \mathfrak{C}$, we define \mathbf{R} -linear mappings $L_a, R_a, T_a : \mathfrak{C} \rightarrow \mathfrak{C}$ respectively by

$$L_ax = ax, \quad R_ax = xa, \quad T_ax = ax + xa = (L_a + R_a)x, \quad x \in \mathfrak{C}.$$

Hereafter, we denote by \mathfrak{C}_0 the subset $\{a \in \mathfrak{C} \mid \bar{a} = -a\}$ of \mathfrak{C} .

Lemma 1.3.3. For $a \in \mathfrak{C}_0$, we have

- (1) $L_a, R_a, T_a \in \mathfrak{D}_4$.
- (2) $\kappa L_a = -R_a, \quad \kappa R_a = -L_a, \quad \kappa T_a = -T_a$.
- (3) $\pi L_a = T_a, \quad \pi R_a = -R_a, \quad \pi T_a = L_a$.
- (4) $\nu L_a = R_a, \quad \nu R_a = -T_a, \quad \nu T_a = -L_a$.

Proof. (1) $(L_ax, y) = (ax, y) = (x, \bar{a}y) = -(x, ay) = -(x, L_ay)$, $x, y \in \mathfrak{C}$. Hence $L_a \in \mathfrak{D}_4$. Similarly, $R_a \in \mathfrak{D}_4$ and $T_a = L_a + R_a \in \mathfrak{D}_4$.

(2) $(\kappa L_a)x = \overline{L_a x} = \overline{ax} = x\bar{a} = -xa = -R_ax$, $x \in \mathfrak{C}$. Hence $\kappa L_a = -R_a$. The others can be similarly obtained.

(3) It is sufficient to show that these relations hold for $a = e_i$, $i = 1, \dots, 7$. We have

$$L_{e_i} = 2F_{i0}, \quad T_{e_i} = 2G_{i0}.$$

Indeed,

$$\begin{aligned}
L_{e_i}x &= e_ix = 2F_{i0}x, \quad x \in \mathfrak{C}, \\
T_{e_i}x &= e_ix + xe_i = \begin{cases} 2e_i, & x = 1 \\ -2, & x = e_i \\ 0, & x = e_j, \quad j \neq 0, i. \end{cases}
\end{aligned}$$

It follows that

$$\pi L_{e_i} = \pi(2F_{i0}) = -2\pi F_{0i} = -2G_{0i} \text{ (Lemma 1.3.1) } = 2G_{i0} = T_{e_i},$$

$$\begin{aligned}\pi T_{e_i} &= \pi(2G_{i0}) = 2F_{i0} = L_{e_i}, \\ \pi R_a &= \pi(T_a - L_a) = L_a - T_a = -R_a.\end{aligned}$$

(4) follows immediately from (2) and (3), since $\nu = \pi\kappa$.

Lemma 1.3.4. *The Lie algebra \mathfrak{D}_4 is generated by $\{L_a \mid a \in \mathfrak{C}_0\}$ (by taking at most one time the Lie bracket $[\ , \]$), that is, any $D \in \mathfrak{D}_4$ is expressed by*

$$D = L_a + \sum_i [L_{b_i}, L_{c_i}], \quad a, b_i, c_i \in \mathfrak{C}_0.$$

Proof. Let \mathfrak{D}' be the Lie subalgebra of \mathfrak{D}_4 generated by $\{L_a \mid a \in \mathfrak{C}_0\}$. Since $L_{e_i} = 2F_{i0}$ and $[L_{e_i}, L_{e_j}] = 4[F_{i0}, F_{j0}] = -4F_{ij}$, $i \neq 0, j \neq 0, i \neq j$, we see that \mathfrak{D}' contains F_{ij} , $i, j = 0, 1, \dots, 7, i \neq j$. Since $\{F_{ij}, i < j\}$ is an \mathbf{R} -basis of \mathfrak{D}_4 (Lemma 1.3.1), \mathfrak{D}' coincides with \mathfrak{D}_4 .

Theorem 1.3.5. *In the automorphism group $\text{Aut}(\mathfrak{D}_4)$ of \mathfrak{D}_4 , the subgroup \mathfrak{S}_3 generated by κ and π is isomorphic to the symmetric group S_3 of degree 3. Furthermore, κ, π, ν have the following relations.*

$$\kappa^2 = 1, \quad \pi^2 = 1, \quad \nu^3 = 1, \quad \nu = \pi\kappa.$$

Proof. Since $\{L_a \mid a \in \mathfrak{C}_0\}$ generates \mathfrak{D}_4 (Lemma 1.3.4), it is sufficient to check $\kappa^2 = 1, \pi^2 = 1, \nu^3 = 1$ etc. for L_a . However, these relations follow from Lemma 1.3.3.(2),(3),(4). The mapping $f : \mathfrak{S}_3 \rightarrow S_3$ defined by the correspondence

$$\begin{aligned}1 &\rightarrow \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \quad \kappa \rightarrow \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, \quad \pi \rightarrow \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}, \\ \nu &\rightarrow \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \quad \nu^2 \rightarrow \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}, \quad \nu\pi \rightarrow \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}\end{aligned}$$

gives an isomorphism as groups.

Theorem 1.3.6. (Principle of infinitesimal triality in \mathfrak{D}_4). *For any $D_1 \in \mathfrak{D}_4$, there exist $D_2, D_3 \in \mathfrak{D}_4$ such that*

$$(D_1 x)y + x(D_2 y) = D_3(xy), \quad x, y \in \mathfrak{C}.$$

Also such D_2, D_3 are uniquely determined for D_1 and we have

$$D_2 = \nu D_1, \quad D_3 = \pi D_1.$$

Proof. If $a \in \mathfrak{C}_0$, then $L_a, R_a, T_a \in \mathfrak{D}_4$ (Lemma 1.3.3.(1)) and the equality $(ax)y + x(ya) = a(xy) + (xy)a$ implies that

$$(L_a x)y + x(R_a y) = T_a(xy), \quad x, y \in \mathfrak{C}. \tag{i}$$

Similarly, for $b \in \mathfrak{C}_0$, we have

$$(L_b x)y + x(R_b y) = T_b(xy), \quad x, y \in \mathfrak{C}. \quad (\text{ii})$$

Applying T_a on (ii) and using (i), we have

$$(L_a L_b x)y + (L_b x)(R_a y) + (L_a x)(R_b y) + x(R_a R_b y) = T_a T_b(xy).$$

Exchanging a for b , and subtracting from the above, we get

$$([L_a, L_b]x)y + x([R_a, R_b]y) = [T_a, T_b](xy). \quad (\text{iii})$$

Since $D_1 \in \mathfrak{D}_4$ is expressed as

$$D_1 = L_a + \sum [L_b, L_c], \quad a, b, c \in \mathfrak{C}_0$$

(Lemma 1.3.4), putting $D_2 = R_a + \sum [R_b, R_c]$, $D_3 = T_a + \sum [T_b, T_c]$ and using (i), (iii), we obtain

$$(D_1 x)y + x(D_2 y) = D_3(xy), \quad x, y \in \mathfrak{C}.$$

Next, we shall show that D_2 and D_3 are determined uniquely for D_1 . To prove this, it is sufficient to show for $D_1 = 0$ that $D_2 = D_3 = 0$. Now, suppose

$$x(D_2 y) = D_3(xy), \quad x, y \in \mathfrak{C}.$$

Putting $x = 1$, we have $D_2 y = D_3 y$, so that $D_2 = D_3 (= D)$. Therefore

$$x(Dy) = D(xy), \quad x, y \in \mathfrak{C}. \quad (\text{iv})$$

Putting $D1 = p$, we have $2(p, 1) = (p, 1) + (1, p) = (D1, 1) + (1, D1) = 0$, which implies $p \in \mathfrak{C}_0$. Furthermore, putting $y = 1$ in (iv), we have $xp = Dx$, so (iv) becomes

$$x(yp) = (xy)p, \quad \text{for all } x, y \in \mathfrak{C}.$$

We therefore see that $p \in \mathbf{R}$, so that $p = 0$ since $p \in \mathfrak{C}_0$. Hence $Dx = xp = 0$, and so $D = 0$. This proves the uniqueness. Finally, if we express $D_1 \in \mathfrak{D}_4$ as $D_1 = L_a + \sum [L_b, L_c]$, $a, b, c \in \mathfrak{C}_0$, then $D_2 = R_a + \sum [R_b, R_c]$, $D_3 = T_a + \sum [T_b, T_c]$ from the arguments above and the uniqueness. Hence we have $D_2 = \nu D_1$, $D_3 = \pi D_1$ (Lemma 1.3.3).

Lemma 1.3.7. *For $D_1, D_2, D_3 \in \mathfrak{D}_4$, the relation*

$$(D_1 x)y + x(D_2 y) = (\kappa D_3)(xy), \quad x, y \in \mathfrak{C}$$

implies

$$(D_2 x)y + x(D_3 y) = (\kappa D_1)(xy), \quad x, y \in \mathfrak{C},$$

$$(D_3 x)y + x(D_1 y) = (\kappa D_2)(xy), \quad x, y \in \mathfrak{C}.$$

Proof. For $D_2 \in \mathfrak{D}_4$, there exist $D_3', D_1' \in \mathfrak{D}_4$ such that

$$(D_2x)y + x(D_3'y) = (\kappa D_1')(xy), \quad x, y \in \mathfrak{C},$$

and $D_3' = \nu D_2$, $\kappa D_1' = \pi D_2$ (Theorem 1.3.6). The assumption of the lemma is $D_2 = \nu D_1$, $\kappa D_3 = \pi D_1$ (Theorem 1.3.6). Hence, using Theorem 1.3.5, we have

$$\begin{aligned} D_3' &= \nu D_2 = \nu \nu D_1 = \nu^{-1} D_1 = \kappa \pi D_1 = \kappa \kappa D_3 = D_3, \\ D_1' &= \kappa \pi D_2 = \nu^{-1} D_2 = D_1. \end{aligned}$$

1.4. Lie algebra \mathfrak{g}_2 of G_2

Theorem 1.4.1. *The Lie algebra \mathfrak{g}_2 of the Lie group G_2 is given by*

$$\mathfrak{g}_2 = \{D \in \text{Hom}_{\mathbf{R}}(\mathfrak{C}) \mid D(xy) = (Dx)y + x(Dy)\}.$$

Proof. If $D \in \text{Hom}_{\mathbf{R}}(\mathfrak{C})$ satisfies $(\exp tD)(xy) = ((\exp tD)x)((\exp tD)y)$, $t \in \mathbf{R}$, then by differentiating with respect to t and putting $t = 0$, we get $D(xy) = (Dx)y + x(Dy)$. Conversely, if $D \in \text{Hom}_{\mathbf{R}}(\mathfrak{C})$ satisfies $D(xy) = (Dx)y + x(Dy)$, then it is not difficult to verify that $\alpha = \exp tD$ satisfies $\alpha(xy) = (\alpha x)(\alpha y)$.

In order to study the Lie algebra \mathfrak{g}_2 , we need the following Lie algebra $\mathfrak{b}_3 = \mathfrak{so}(7)$ of $SO(7)$:

$$\begin{aligned} \mathfrak{b}_3 &= \{D \in \mathfrak{D}_4 \mid D1 = 0\} \\ &= \{D \in \mathfrak{D}_4 \mid \kappa D = D\} = \{D \in \mathfrak{D}_4 \mid \nu D = \pi D\}. \end{aligned}$$

Lemma 1.4.2. \mathfrak{g}_2 is a Lie subalgebra of \mathfrak{b}_3 . Moreover we have

$$\begin{aligned} \mathfrak{g}_2 &= \{D \in \mathfrak{D}_4 \mid \nu D = D, \pi D = D\} \\ &= \{D \in \mathfrak{D}_4 \mid \lambda D = D, \lambda \in \mathfrak{S}_3\} \\ &= \{D \in \mathfrak{b}_3 \mid \pi D = D\}. \end{aligned}$$

Proof. Let $D \in \mathfrak{g}_2$. Putting $x = y = 1$ in $D(xy) = (Dx)y + x(Dy)$, we get $D1 = 0$. We next show that

$$Dx \in \mathfrak{C}_0, \quad x \in \mathfrak{C}.$$

If $i \neq 0$, then $(De_i)e_i + e_i(De_i) = D(e_ie_i) = D(-1) = 0$, and so $De_i \in \mathfrak{C}_0$. Together with $D1 = 0$, we have $Dx \in \mathfrak{C}_0$, $x \in \mathfrak{C}$. Now, note that

$$xy + yx = -(x\bar{y} + y\bar{x}) = -2(x, y), \quad x, y \in \mathfrak{C}_0.$$

Applying D on the relation above, we have $(Dx)y + x(Dy) + (Dy)x + y(Dx) = 0$. Since $Dx, Dy \in \mathfrak{C}_0$, we get

$$(Dx, y) + (x, Dy) = 0, \quad x, y \in \mathfrak{C}.$$

Hence $\mathfrak{g}_2 \subset \mathfrak{b}_3$. The relation in the lemma follows easily from Theorem 1.3.5 and Theorem 1.3.6.

Theorem 1.4.3. *Any element of \mathfrak{g}_2 is expressed by the sum of elements of the following seven types.*

$$\begin{aligned} \lambda G_{23} + \mu G_{45} + \nu G_{67}, & \quad -\lambda G_{13} - \mu G_{46} + \nu G_{57}, \\ \lambda G_{12} + \mu G_{47} + \nu G_{56}, & \quad -\lambda G_{15} + \mu G_{26} - \nu G_{37}, \quad \lambda, \mu, \nu \in \mathbf{R} \\ \lambda G_{14} - \mu G_{27} - \nu G_{36}, & \quad -\lambda G_{17} - \mu G_{24} + \nu G_{35}, \quad \lambda + \mu + \nu = 0 \\ & \quad \lambda G_{16} + \mu G_{25} + \nu G_{34}. \end{aligned}$$

Conversely, the above seven elements belong to \mathfrak{g}_2 . In particular, the dimension of \mathfrak{g}_2 is 14:

$$\dim \mathfrak{g}_2 = 14.$$

Proof. Let $D \in \mathfrak{g}_2$. Since $D \in \mathfrak{b}_3$ (Lemma 1.4.2), $D = \sum_{0 < i < j} \lambda_{ij} G_{ij}$, $\lambda_{ij} \in \mathbf{R}$. The condition $\pi D = D$ (Lemma 1.4.2) implies that

$$\sum_{0 < i < j} \lambda_{ij} F_{ij} = \sum_{0 < i < j} \lambda_{ij} G_{ij}.$$

Applying the above element on $1 \in \mathfrak{C}$, we have

$$\sum_{0 < i < j} \lambda_{ij} e_j e_i = 0.$$

Replacing $e_j e_i$ by e_k and comparing the coefficient of each e_1, e_2, \dots, e_7 , we have

$$\begin{aligned} \lambda_{23} + \lambda_{45} + \lambda_{67} &= 0, & -\lambda_{13} - \lambda_{46} + \lambda_{57} &= 0, \\ \lambda_{12} + \lambda_{47} + \lambda_{56} &= 0, & -\lambda_{15} + \lambda_{26} - \lambda_{37} &= 0, \\ \lambda_{14} - \lambda_{27} - \lambda_{36} &= 0, & -\lambda_{17} - \lambda_{24} + \lambda_{35} &= 0, \\ \lambda_{16} + \lambda_{25} + \lambda_{34} &= 0, \end{aligned}$$

from which the first result follows. Conversely, any of these seven elements D obviously belongs to \mathfrak{b}_3 and the condition $\pi D = D$ (Lemma 1.4.2) is verified from the table of Lemma 1.3.1.

1.5. Lie subalgebra $\mathfrak{su}(3)$ of \mathfrak{g}_2

The Cayley algebra \mathfrak{C} naturally contains the field \mathbf{C} of complex numbers as

$$\mathbf{C} = \{x_0 + x_1 e_1 \mid x_i \in \mathbf{R}\}.$$

Any element $x \in \mathfrak{C}$ is expressed by

$$\begin{aligned} x &= x_0 + x_1 e_1 + x_2 e_2 + x_3 e_3 + x_4 e_4 + x_5 e_5 + x_6 e_6 + x_7 e_7 \quad (x_i \in \mathbf{R}) \\ &= (x_0 + x_1 e_1) + (x_2 + x_3 e_1) e_2 + (x_4 + x_5 e_1) e_4 + (x_6 + x_7 e_1) e_6, \end{aligned}$$

that is,

$$x = a + m_1 e_2 + m_2 e_4 + m_3 e_6, \quad a, m_i \in \mathcal{C}.$$

We associate such element x of \mathfrak{C} with the element of $\mathcal{C} \oplus \mathcal{C}^3$

$$a + \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix}.$$

In $\mathcal{C} \oplus \mathcal{C}^3$, we define a multiplication, an inner product $(\ , \)$ and a conjugation $\overline{\ \ }$ respectively by

$$\begin{aligned} (a + \mathbf{m})(b + \mathbf{n}) &= (ab - \langle \mathbf{m}, \mathbf{n} \rangle) + (a\mathbf{n} + \bar{b}\mathbf{m} - \overline{\mathbf{m} \times \mathbf{n}}), \\ (a + \mathbf{m}, b + \mathbf{n}) &= (a, b) + (\mathbf{m}, \mathbf{n}), \\ \overline{a + \mathbf{m}} &= \bar{a} - \mathbf{m}, \end{aligned}$$

where the real valued symmetric inner product (\mathbf{m}, \mathbf{n}) , the Hermitian inner product $\langle \mathbf{m}, \mathbf{n} \rangle$ and the exterior product $\mathbf{m} \times \mathbf{n}$ are usually defined respectively by

$$\begin{aligned} (\mathbf{m}, \mathbf{n}) &= \frac{1}{2}(\mathbf{m}^* \mathbf{n} + \mathbf{n}^* \mathbf{m}) = \sum_{i=1}^3 (m_i, n_i), \quad \langle \mathbf{m}, \mathbf{n} \rangle = \sum_{i=1}^3 m_i \bar{n}_i, \\ \mathbf{m} \times \mathbf{n} &= \begin{pmatrix} m_2 n_3 - n_2 m_3 \\ m_3 n_1 - n_3 m_1 \\ m_1 n_2 - n_1 m_2 \end{pmatrix} \end{aligned}$$

for $\mathbf{m} = \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix}$, $\mathbf{n} = \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} \in \mathcal{C}^3$. Since these operations correspond to their respective operations in \mathfrak{C} , hereafter, we identify $\mathcal{C} \oplus \mathcal{C}^3$ with \mathfrak{C} , that is,

$$\mathcal{C} \oplus \mathcal{C}^3 = \mathfrak{C}.$$

We shall study the following subalgebra $(\mathfrak{g}_2)_{e_1}$ of \mathfrak{g}_2 :

$$(\mathfrak{g}_2)_{e_1} = \{D \in \mathfrak{g}_2 \mid D e_1 = 0\}.$$

Theorem 1.5.1. $(\mathfrak{g}_2)_{e_1} \cong \mathfrak{su}(3)$.

Proof. We define a mapping $\varphi_* : \mathfrak{su}(3) \rightarrow \{D \in M(3, \mathcal{C}) \mid D^* = -D, \text{tr}(D) = 0\} \rightarrow (\mathfrak{g}_2)_{e_1}$ by

$$\varphi_*(D)(a + \mathbf{m}) = D\mathbf{m}, \quad a + \mathbf{m} \in \mathcal{C} \oplus \mathcal{C}^3 = \mathfrak{C}.$$

We first prove that $\varphi_*(D) \in (\mathfrak{g}_2)_{e_1}$. For elements

$$\begin{aligned} &e_1(E_{11} - E_{22}), \quad e_1(E_{22} - E_{33}), \quad E_{12} - E_{21}, \quad e_1(E_{12} + E_{21}), \\ &E_{13} - E_{31}, \quad e_1(E_{13} + E_{31}), \quad E_{23} - E_{32}, \quad e_1(E_{23} + E_{32}) \end{aligned}$$

of an \mathbf{R} -basis of $\mathfrak{su}(3)$ (where $E_{kl} \in M(3, \mathbf{R})$ is the matrix with the (k, l) -entry is 1 and otherwise are 0), we have

$$\begin{aligned}\varphi_*(e_1(E_{11} - E_{22})) &= -G_{23} + G_{45}, & \varphi_*(e_1(E_{22} - E_{33})) &= -G_{45} + G_{67}, \\ \varphi_*(E_{12} - E_{21}) &= G_{24} + G_{35}, & \varphi_*(e_1(E_{12} + E_{21})) &= -G_{25} + G_{34}, \\ \varphi_*(E_{13} - E_{31}) &= G_{26} + G_{37}, & \varphi_*(e_1(E_{13} + E_{31})) &= -G_{27} + G_{36}, \\ \varphi_*(E_{23} - E_{32}) &= G_{46} + G_{57}, & \varphi_*(e_1(E_{23} + E_{32})) &= -G_{47} + G_{56}.\end{aligned}$$

Hence $\varphi_*(D) \subset \mathfrak{g}_2$ (Theorem 1.4.3). Clearly $\varphi_*(D)e_1 = 0$, so that $\varphi_*(D) \subset (\mathfrak{g}_2)_{e_1}$. Obviously $\varphi_* : \mathfrak{su}(3) \rightarrow \mathfrak{g}_2$ is a homomorphism as Lie algebras and is injective, so that we identify $\mathfrak{su}(3)$ and $\varphi_*(\mathfrak{su}(3))$. We set

$$\begin{aligned}S_1 &= 2G_{12} - G_{47} - G_{56}, & S_2 &= 2G_{13} - G_{46} + G_{57}, \\ S_3 &= 2G_{14} + G_{27} + G_{36}, & S_4 &= 2G_{15} + G_{26} - G_{37}, \\ S_5 &= 2G_{16} - G_{25} - G_{34}, & S_6 &= 2G_{17} - G_{24} + G_{35},\end{aligned}$$

and let \mathfrak{S} be the \mathbf{R} -vector subspace of \mathfrak{g}_2 spanned by S_1, \dots, S_6 . Then we have the following decomposition of \mathfrak{g}_2 .

$$\mathfrak{g}_2 = \mathfrak{su}(3) \oplus \mathfrak{S}.$$

Now, we shall show that $\varphi_* : \mathfrak{su}(3) \rightarrow (\mathfrak{g}_2)_{e_1}$ is onto. Let $B \in (\mathfrak{g}_2)_{e_1}$. Denote

$$B = D + \sum_{i=1}^6 x_i S_i, \quad D \in \mathfrak{su}(3), x_i \in \mathbf{R}.$$

From the condition $Be_1 = 0$, we have

$$-2x_1e_2 - 2x_2e_3 - 2x_3e_4 - 2x_4e_5 - 2x_5e_6 - 2x_6e_7 = 0.$$

Hence $x_1 = \dots = x_6 = 0$, so that $B = D \in \mathfrak{su}(3)$. Thus the proof of Theorem 1.5.1 is completed.

1.6. Simplicity of \mathfrak{g}_2^C

Let $\mathfrak{C}^C = \{x_1 + ix_2 \mid x_1, x_2 \in \mathfrak{C}\}$ be the complexification of the Cayley algebra \mathfrak{C} . In the same manner as in \mathfrak{C} , we can also define in \mathfrak{C}^C the multiplication xy , the inner product (x, y) such that they satisfy properties 1 ~ 12.3 of Section 1.1 (except some formulas about the length $|x|$). \mathfrak{C}^C is called the complex Cayley algebra. \mathfrak{C}^C has two complex conjugations, namely,

$$\overline{x_1 + ix_2} = \overline{x_1} + i\overline{x_2}, \quad \tau(x_1 + ix_2) = x_1 - ix_2, \quad x_i \in \mathfrak{C}.$$

The complex conjugation τ is a complex-conjugate linear transformation of \mathfrak{C}^C and satisfies

$$\tau(xy) = (\tau x)(\tau y), \quad x, y \in \mathfrak{C}^C.$$

For a while, in the Lie algebra $\mathfrak{su}(3) \subset \mathfrak{g}_2$, we use the following notations.

$$\begin{aligned} H_1 &= -G_{23} + G_{45}, & H_2 &= -G_{45} + G_{67} \\ L_{12} &= G_{24} + G_{35}, & L_{21} &= -G_{25} + G_{34}, & L_{13} &= G_{26} + G_{37}, \\ L_{21} &= -G_{27} + G_{36}, & L_{23} &= G_{46} + G_{57}, & L_{32} &= -G_{47} + G_{56}. \end{aligned}$$

Lemma 1.6.1. *The Lie brackets $[D, S], D \in \mathfrak{su}(3), S \in \mathfrak{S}$ are given as follows.*

	S_1	S_2	S_3	S_4	S_5	S_6
H_1	S_2	$-S_1$	$-S_4$	S_3	0	0
H_2	0	0	S_4	$-S_3$	$-S_6$	S_5
L_{12}	$-S_3$	$-S_4$	S_1	S_2	0	0
L_{21}	S_4	$-S_3$	S_2	$-S_1$	0	0
L_{13}	$-S_5$	$-S_6$	0	0	S_1	S_2
L_{31}	S_6	$-S_5$	0	0	S_2	$-S_1$
L_{23}	0	0	$-S_5$	$-S_6$	S_3	S_4
L_{32}	0	0	S_6	$-S_5$	S_4	$-S_3$

Lemma 1.6.2. *\mathfrak{S} is a $\mathfrak{su}(3)$ -irreducible \mathbf{R} -module, and hence we have*

$$[\mathfrak{su}(3), \mathfrak{S}] = \mathfrak{S}.$$

Proof. Evidently \mathfrak{S} is a $\mathfrak{su}(3)$ - \mathbf{R} -module. Let W be a non-zero $\mathfrak{su}(3)$ -invariant \mathbf{R} -submodule of \mathfrak{S} . If W contains some S_k , then, from the table of Lemma 1.6.1, we can see that W contains all $S_k, k = 1, 2, \dots, 6$, and hence $W = \mathfrak{S}$. Now, let $S = \sum_{k=1}^6 x_k S_k, x_k \in \mathbf{R}$ be a non-zero element of W and assume that $x_1 \neq 0$ (without the loss of generality). Applying H_1 on S , we have $x_1 S_2 - x_2 S_1 - x_3 S_4 + x_4 S_3 \in W$. Next, applying L_{13} and L_{31} on it, we have

$$-x_1 S_6 + x_2 S_5 \in W \quad \dots (i) \quad \text{and} \quad -x_1 S_5 - x_2 S_6 \in W \quad \dots (ii)$$

Taking (ii) $\times x_2 -$ (i) $\times x_1$, we have $(x_1^2 + x_2^2) S_5 \in W$. Since $x_1^2 + x_2^2 \neq 0$, we have $S_5 \in W$ and so $W = \mathfrak{S}$. Consequently the irreducibility of \mathfrak{S} is proved. Finally, since $[\mathfrak{su}(3), \mathfrak{S}]$ is a $\mathfrak{su}(3)$ -invariant \mathbf{R} -submodule of \mathfrak{S} , from the irreducibility of \mathfrak{S} , we have $[\mathfrak{su}(3), \mathfrak{S}] = \mathfrak{S}$.

Theorem 1.6.3. *The Lie algebra \mathfrak{g}_2^C is simple and so \mathfrak{g}_2 is also simple.*

Proof. We shall prove that \mathfrak{g}_2 is simple, because the proof of that \mathfrak{g}_2^C is simple is the same as the case of \mathfrak{g}_2 . We use the decomposition of \mathfrak{g}_2

$$\mathfrak{g}_2 = \mathfrak{su}(3) \oplus \mathfrak{S} \quad (\text{Theorem 1.5.1}).$$

Let $p : \mathfrak{g}_2 \rightarrow \mathfrak{su}(3)$ and $q : \mathfrak{g}_2 \rightarrow \mathfrak{S}$ be projections of $\mathfrak{g}_2 = \mathfrak{su}(3) \oplus \mathfrak{S}$. Now, let \mathfrak{a} be a non-zero ideal of \mathfrak{g}_2 . Then $p(\mathfrak{a})$ is an ideal of \mathfrak{g}_2 . Indeed, if $D \in p(\mathfrak{a})$, then there exists $S \in \mathfrak{S}$ such that $D + S \in \mathfrak{a}$. For any $D' \in \mathfrak{su}(3)$, we have

$$\mathfrak{a} \ni [D', D + S] = [D', D] + [D', S], \quad [D', S] \in \mathfrak{S}$$

(Lemma 1.6.1), hence $[D', D] \in p(\mathfrak{a})$.

We show that either $\mathfrak{su}(3) \cap \mathfrak{a} \neq \{0\}$ or $\mathfrak{S} \cap \mathfrak{a} \neq \{0\}$. Assume that $\mathfrak{su}(3) \cap \mathfrak{a} = \{0\}$ and $\mathfrak{S} \cap \mathfrak{a} = \{0\}$. The mapping $p|_{\mathfrak{a}} : \mathfrak{a} \rightarrow \mathfrak{su}(3)$ is injective because $\mathfrak{S} \cap \mathfrak{a} = \{0\}$. Since $p(\mathfrak{a})$ is a non-zero ideal of $\mathfrak{su}(3)$ and $\mathfrak{su}(3)$ is simple, we have $p(\mathfrak{a}) = \mathfrak{su}(3)$. Hence $\dim \mathfrak{a} = \dim p(\mathfrak{a}) = \dim \mathfrak{su}(3) = 8$. On the other hand, since $\mathfrak{su}(3) \cap \mathfrak{a} = \{0\}$, $q|_{\mathfrak{a}} : \mathfrak{a} \rightarrow \mathfrak{S}$ is also injective, we have $\dim \mathfrak{a} \leq \dim \mathfrak{S} = 6$. This leads to a contradiction.

We now consider the following two cases.

(1) Case $\mathfrak{su}(3) \cap \mathfrak{a} \neq \{0\}$. From the simplicity of $\mathfrak{su}(3)$, we have $\mathfrak{su}(3) \cap \mathfrak{a} = \mathfrak{su}(3)$, hence $\mathfrak{a} \supset \mathfrak{su}(3)$. On the other hand, we have

$$\mathfrak{a} \supset [\mathfrak{a}, \mathfrak{S}] \supset [\mathfrak{su}(3), \mathfrak{S}] = \mathfrak{S} \quad (\text{Lemma 1.6.1}).$$

Hence $\mathfrak{a} \supset \mathfrak{su}(3) \oplus \mathfrak{S} = \mathfrak{g}_2$.

(2) Case $\mathfrak{S} \cap \mathfrak{a} \neq \{0\}$. Choose a non-zero element $S \in \mathfrak{S} \cap \mathfrak{a} \subset \mathfrak{a}$. Under the actions of $\mathfrak{su}(3)$, we can see that $S_1 \in \mathfrak{a}$ (Lemma 1.6.1). Hence $0 \neq 4H_1 + 2H_2 = [S_1, S_2] \in \mathfrak{a}$. So this case can be reduced to the case (1). Thus we have $\mathfrak{a} = \mathfrak{g}_2$, which proves the simplicity of \mathfrak{g}_2 .

1.7. Killing form of \mathfrak{g}_2^C

Lemma 1.7.1. *In $\mathfrak{su}(3) \subset \mathfrak{g}_2$, the Lie brackets between H_1 and L_{ij}, L_{ji} of Section 1.6 are given by*

$$\begin{aligned} [H_1, L_{12}] &= 2L_{21}, & [H_1, L_{21}] &= -2L_{12}, & [H_1, L_{13}] &= L_{31}, \\ [H_1, L_{31}] &= -L_{13}, & [H_1, L_{23}] &= -L_{32}, & [H_1, L_{32}] &= L_{23}. \end{aligned}$$

Theorem 1.7.2. *The killing form B_2 of the Lie algebra \mathfrak{g}_2^C is given by*

$$B_2(D_1, D_2) = 4\text{tr}(D_1 D_2), \quad D_i \in \mathfrak{g}_2^C.$$

Proof. Since $\text{tr}(D_1 D_2)$ is a \mathfrak{g}_2^C -adjoint invariant bilinear form of \mathfrak{g}_2^C and \mathfrak{g}_2^C is simple (Theorem 1.6.3), there exists $k \in C$ such that

$$B_2(D_1, D_2) = k\text{tr}(D_1 D_2).$$

To determine this k , let $D_1 = D_2 = H_1$. Then from

$$\begin{aligned} [H_1, [H_1, L_{12}]] &= [H_1, 2L_{21}] = -4L_{12}, & [H_1, [H_1, L_{21}]] &= [H_1, -2L_{12}] = -4L_{21}, \\ [H_1, [H_1, L_{13}]] &= [H_1, L_{31}] = -L_{13}, & [H_1, [H_1, L_{31}]] &= [H_1, -L_{13}] = -L_{31}, \\ [H_1, [H_1, L_{23}]] &= [H_1, -L_{32}] = -L_{23}, & [H_1, [H_1, L_{32}]] &= [H_1, L_{23}] = -L_{32}, \end{aligned}$$

$$\begin{aligned} [H_1, [H_1, S_1]] &= [H_1, S_2] = -S_1, & [H_1, [H_1, S_2]] &= [H_1, -S_1] = -S_2, \\ [H_1, [H_1, S_3]] &= [H_1, -S_4] = -S_3, & [H_1, [H_1, S_4]] &= [H_1, S_3] = -S_4, \\ [H_1, [H_1, S_5]] &= 0, & [H_1, [H_1, S_6]] &= 0 \end{aligned}$$

(Lemma 1.6.1), we have

$$B_2(H_1, H_1) = \text{tr}((\text{ad}H_1)^2) = (-4) \times 2 + (-1) \times 8 = -16.$$

On the other hand,

$$\begin{aligned} H_1 H_1 e_2 &= H_1 e_3 = -e_2, & H_1 H_1 e_3 &= -H_1 e_2 = -e_3, \\ H_1 H_1 e_4 &= -H_1 e_5 = -e_4, & H_1 H_1 e_5 &= H_1 e_4 = -e_4, \\ H_1 H_1 e_i &= 0 \text{ otherwise.} \end{aligned}$$

Hence $\text{tr}(H_1 H_1) = (-1) \times 4 = -4$. Therefore $k = 4$.

1.8. Roots of \mathfrak{g}_2^C

We recall the C -Lie isomorphism $f_* : \mathfrak{sl}(3, C) \rightarrow \mathfrak{su}(3)^C$ and the embedding $\varphi_* : \mathfrak{su}(3)^C \rightarrow \mathfrak{g}_2^C$,

$$\begin{aligned} f_*(A) &= \varepsilon A - \bar{\varepsilon}^t A, & \varepsilon &= \frac{1}{2}(1 + ie_1), \\ \varphi_*(D)(a + \mathbf{m}) &= D\mathbf{m}, & a + \mathbf{m} &\in \mathbf{C}^C \oplus (\mathbf{C}^3)^C = \mathfrak{C}^C, \end{aligned}$$

and we regard $\mathfrak{sl}(3, C)$ as a subalgebra of \mathfrak{g}_2^C under the composition of f_* and φ_* . Further we know that the Lie algebra $\mathfrak{sl}(3, C)$ has roots $\pm(\lambda_k - \lambda_l)$, $1 \leq k < l \leq 3$ relative to the Cartan subalgebra

$$\mathfrak{h} = \left\{ \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \mid \lambda_i \in \mathbf{C}, \lambda_1 + \lambda_2 + \lambda_3 = 0 \right\}$$

of $\mathfrak{sl}(3, C)$, and E_{kl} is a root vector associated with the root $\lambda_k - \lambda_l$.

Theorem 1.8.1. *The rank of the Lie algebra \mathfrak{g}_2^C is 2. The roots of \mathfrak{g}_2^C relative to some Cartan subalgebra of \mathfrak{g}_2^C are given by*

$$\pm(\lambda_1 - \lambda_2), \quad \pm(\lambda_1 - \lambda_3), \quad \pm(\lambda_2 - \lambda_3), \quad \pm\lambda_1, \quad \pm\lambda_2, \quad \pm\lambda_3$$

with $\lambda_1 + \lambda_2 + \lambda_3 = 0$.

Proof. $\mathfrak{h} = \{-i\lambda_1 G_{23} - i\lambda_2 G_{45} - i\lambda_3 G_{67} \in \mathfrak{g}_2^C \mid \lambda_k \in C\} \subset \mathfrak{sl}(3, C) \subset \mathfrak{g}_2^C$ is an abelian subalgebra of \mathfrak{g}_2^C (it will be a Cartan subalgebra of \mathfrak{g}_2^C). The roots of $\mathfrak{sl}(3, C)$ is also roots of \mathfrak{g}_2^C , so we have the table of roots and associated root vectors as follows.

$$\begin{aligned} \pm(\lambda_1 - \lambda_2) &: \pm(G_{24} + G_{35}) + i(-G_{25} + G_{34}), \\ \pm(\lambda_1 - \lambda_3) &: \pm(G_{26} + G_{37}) + i(-G_{27} + G_{36}), \\ \pm(\lambda_2 - \lambda_3) &: \pm(G_{46} + G_{57}) + i(-G_{47} + G_{56}). \end{aligned}$$

The remainder roots and associated root vectors are found as follows.

$$\begin{aligned}\pm\lambda_1 &: (2G_{12} - G_{47} - G_{56}) \pm i(2G_{13} - G_{46} + G_{57}), \\ \pm\lambda_2 &: (2G_{14} + G_{27} + G_{36}) \pm i(2G_{15} + G_{26} - G_{37}), \\ \pm\lambda_3 &: (2G_{16} - G_{25} - G_{34}) \pm i(2G_{17} - G_{24} + G_{35}).\end{aligned}$$

Theorem 1.8.2. *In the root system of \mathfrak{g}_2^C of Theorem 1.8.1,*

$$\alpha_1 = \lambda_1 - \lambda_2, \quad \alpha_2 = \lambda_2$$

is a fundamental root system of the Lie algebra \mathfrak{g}_2^C and

$$\mu = 2\alpha_1 + 3\alpha_3$$

is the highest root. The Dynkin diagram and the extended Dynkin diagram of \mathfrak{g}_2^C are respectively given by



Proof. All positive roots of \mathfrak{g}_2^C are expressed by

$$\begin{aligned}\lambda_1 - \lambda_2 &= \alpha_1, & \lambda_1 - \lambda_3 &= 2\alpha_1 + 3\alpha_2, & \lambda_2 - \lambda_3 &= \alpha_1 + 3\alpha_2, \\ \lambda_1 &= \alpha_1 + \alpha_2, & \lambda_2 &= \alpha_2, & -\lambda_3 &= \alpha_1 + 2\alpha_2.\end{aligned}$$

Hence $\Pi = \{\alpha_1, \alpha_2\}$ is a fundamental root system of \mathfrak{g}_2^C . The real part of $\mathfrak{h}_{\mathbf{R}}$ of \mathfrak{h} is

$$\mathfrak{h}_{\mathbf{R}} = \{-i\lambda_1 G_{23} - i\lambda_2 G_{45} - i\lambda_3 G_{67} \mid \lambda_i \in \mathbf{R}, \lambda_1 + \lambda_2 + \lambda_3 = 0\}$$

and the Killing form B_2 on $\mathfrak{h}_{\mathbf{R}}$ is given by

$$B_2(H, H') = 8 \sum_{k=1}^3 \lambda_k \lambda_k',$$

for $H = -i\lambda_1 G_{23} - i\lambda_2 G_{45} - i\lambda_3 G_{67}$, $H' = -i\lambda_1' G_{23} - i\lambda_2' G_{45} - i\lambda_3' G_{67} \in \mathfrak{h}_{\mathbf{R}}$ (Theorem 1.7.2). Now, the canonical element $H_{\alpha_i} \in \mathfrak{h}_{\mathbf{R}}$ associated with α_i ($B_2(H_{\alpha_i}, H) = \alpha_i(H)$, $H \in \mathfrak{h}_{\mathbf{R}}$) are determined as follows.

$$H_{\alpha_1} = -\frac{1}{8}iG_{23} + \frac{1}{8}iG_{45}, \quad H_{\alpha_2} = \frac{1}{24}iG_{23} - \frac{1}{12}iG_{45} + \frac{1}{24}iG_{67}.$$

Hence we have

$$\begin{aligned}(\alpha_1, \alpha_1) &= B_2(H_{\alpha_1}, H_{\alpha_1}) = 8 \frac{1}{8} \frac{1}{8} (1 + 1) = \frac{1}{4}, \\ (\alpha_2, \alpha_2) &= B_2(H_{\alpha_2}, H_{\alpha_2}) = 8 \frac{1}{24} \frac{1}{24} (1 + 4 + 1) = \frac{1}{12}, \\ (\alpha_1, \alpha_2) &= B_2(H_{\alpha_1}, H_{\alpha_2}) = 8 \frac{1}{8} \frac{1}{24} (-1 - 2) = -\frac{1}{8}, \\ (-\mu, -\mu) &= \frac{1}{4}, \quad (-\mu, \alpha_1) = -\frac{1}{8}, \quad (-\mu, \alpha_2) = 0.\end{aligned}$$

Using them, we can draw the Dynkin diagram and the extended Dynkin diagram of \mathfrak{g}_2^C .

According to Borel-Siebenthal theory (Borel and Siebenthal [4]), the Lie algebra \mathfrak{g}_2 has two subalgebras as maximal subalgebras with the maximal rank 2.

(1) One is a subalgebra of type $C_1 \oplus C_1$ which is obtained as the fixed points by an involution γ of \mathfrak{g}_2 .

(2) The other is a subalgebra of type A_2 which is obtained as the fixed points by an automorphism w of order 3 of \mathfrak{g}_2 .

These subalgebras will be realized in the group G_2 in Theorem 1.10.1 and Theorem 1.9.4, respectively.

1.9. Automorphism w of order 3 and subgroup $SU(3)$ of G_2

We shall study the following subgroup $(G_2)_{e_1}$ of G_2 :

$$(G_2)_{e_1} = \{\alpha \in G_2 \mid \alpha e_1 = e_1\}.$$

Theorem 1.9.1. $(G_2)_{e_1} \cong SU(3)$.

Proof (cf. Theorem 1.5.1). We define a mapping $\varphi : SU(3) \rightarrow (G_2)_{e_1}$ by

$$\varphi(A)(a + \mathbf{m}) = a + A\mathbf{m}, \quad a + \mathbf{m} \in \mathbf{C} \oplus \mathbf{C}^3 = \mathfrak{C}.$$

We first prove that $\varphi(A) \in (G_2)_{e_1}$. For $\alpha = \varphi(A)$, $A \in SU(3)$ and $x = a + \mathbf{m}$, $y = b + \mathbf{n} \in \mathbf{C} \oplus \mathbf{C}^3 = \mathfrak{C}$, using that if $A \in SU(3)$ then \tilde{A} (which is the adjoint matrix of A) $A = A^{-1} = A^*$, we have

$$\begin{aligned} (\alpha x)(\alpha y) &= (a + A\mathbf{m})(b + A\mathbf{n}) \\ &= (ab - \langle A\mathbf{m}, A\mathbf{n} \rangle) + (aA\mathbf{n} + \bar{b}A\mathbf{m} - \overline{A\mathbf{m} \times A\mathbf{n}}) \\ &= (ab - \langle \mathbf{m}, A^*A\mathbf{n} \rangle) + (aA\mathbf{n} + \bar{b}A\mathbf{m} - \overline{{}^t\tilde{A}(\mathbf{m} \times \mathbf{n})}) \\ &= (ab - \langle \mathbf{m}, \mathbf{n} \rangle) + A(a\mathbf{n} + \bar{b}\mathbf{m} - \overline{\mathbf{m} \times \mathbf{n}}) \\ &= \varphi(A)((a + \mathbf{m})(b + \mathbf{n})) = \alpha(xy). \end{aligned}$$

Hence $\varphi(A) \in G_2$. Clearly $\varphi(A)e_1 = e_1$, so that $\varphi(A) \in (G_2)_{e_1}$. Evidently φ is a homomorphism. We show that φ is onto. Let $\alpha \in (G_2)_{e_1}$. Note that α induces a \mathbf{C} -linear transformation of \mathbf{C}^3 . Now let

$$\alpha e_2 = \mathbf{a}_1, \quad \alpha e_4 = \mathbf{a}_2, \quad \alpha e_6 = \mathbf{a}_3$$

and construct a matrix $A = (\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3) \in M(3, \mathbf{C})$. From $(\alpha e_2)(\alpha e_4) = \alpha(e_2 e_4) = -\alpha e_6$, we have $\mathbf{a}_1 \mathbf{a}_2 = -\mathbf{a}_3$, namely, $-\langle \mathbf{a}_1, \mathbf{a}_2 \rangle - \overline{\mathbf{a}_1} \times \overline{\mathbf{a}_2} = -\mathbf{a}_3$, hence we have

$$\langle \mathbf{a}_1, \mathbf{a}_2 \rangle = 0, \quad \mathbf{a}_3 = \overline{\mathbf{a}_1} \times \overline{\mathbf{a}_2}.$$

Similarly we have $\langle \mathbf{a}_2, \mathbf{a}_3 \rangle = \langle \mathbf{a}_3, \mathbf{a}_1 \rangle = 0$. Moreover from $(\alpha e_k)(\alpha e_k) = \alpha(e_k e_k) = \alpha(-1) = -1$, we have $\langle \mathbf{a}_k, \mathbf{a}_k \rangle = 1$, hence $A \in U(3)$. Finally $\det A = (\mathbf{a}_3, \mathbf{a}_1 \times \mathbf{a}_2) = (\mathbf{a}_3, \bar{\mathbf{a}}_3) = \langle \mathbf{a}_3, \mathbf{a}_3 \rangle = 1$ (where (\mathbf{a}, \mathbf{b}) is the ordinary inner product in \mathcal{C}^3 : $(\mathbf{a}, \mathbf{b}) = {}^t \mathbf{a} \mathbf{b}$). Hence we have $A \in SU(3)$ and $\varphi(A) = \alpha$, which shows that φ is onto. $\text{Ker } \varphi = \{E\}$ is easily obtained. Thus we have the isomorphism $SU(3) \cong (G_2)_{e_1}$.

Theorem 1.9.2. $G_2/SU(3) \simeq S^6$.

Proof. $S^6 = \{a \in \mathfrak{C} \mid \bar{a} = -a, |a| = 1\}$ is a 6 dimensional sphere. Since the group G_2 is a subgroup of $O(7) = \{\alpha \in O(\mathfrak{C}) \mid \alpha 1 = 1\}$ (Remark of Theorem 1.2.2), G_2 acts on S^6 . We shall show that this action is transitive. To prove this, it is sufficient to show that any element $a \in S^6$ can be transformed to $e_1 \in S^6$ by some $\alpha \in G_2$. Now, for $a_1 \in S^6$, choose any element $a_2 \in S^6$ such that $(a_1, a_2) = 0$. Let

$$a_3 = a_1 a_2.$$

Then $a_3 \in S^6$ and a_3 satisfies $(a_1, a_3) = (a_2, a_3) = 0$. Choose any element $a_4 \in S^6$ such that $(a_1, a_4) = (a_2, a_4) = (a_3, a_4) = 0$. Let

$$a_5 = a_1 a_4, \quad a_6 = a_4 a_2, \quad a_7 = a_3 a_4.$$

Then the set $\{a_0 = 1, a_1, a_2, \dots, a_7\}$ is an orthonormal \mathbf{R} -basis of \mathfrak{C} . Indeed, $|a_i| = 1, 0 \leq i \leq 7$ are trivial, and we need to verify that $(a_i, a_j) = 0, i \neq j$. However this can be checked by direct calculations such as

$$\begin{aligned} (a_4, a_7) &= (a_4, a_3 a_4) = (1, a_3)(a_4, a_4) = 0, \\ (a_1, a_6) &= (a_1, a_4 a_2) = -(a_1 a_2, a_4) = -(a_3, a_4) = 0, \\ (a_3, a_6) &= (a_3, a_4 a_2) = -(a_3 a_2, a_4) = (a_1, a_4) = 0, \text{ etc.} \end{aligned}$$

Now, since $\{e_0 = 1, e_1, e_2, \dots, e_7\}$ and $\{a_0 = 1, a_1, a_2, \dots, a_7\}$ are both orthonormal \mathbf{R} -bases, the \mathbf{R} -linear isomorphism $\alpha : \mathfrak{C} \rightarrow \mathfrak{C}$ satisfying

$$\alpha e_i = a_i, \quad i = 0, 1, \dots, 7$$

belongs to $O(7)$: $\alpha \in O(7)$. Moreover we claim that $\alpha \in G_2$, that is, α satisfies

$$\alpha(xy) = (\alpha x)(\alpha y), \quad x, y \in \mathfrak{C}.$$

To show this, it is sufficient to note that

$$\alpha(e_i e_j) = (\alpha e_i)(\alpha e_j), \quad i, j = 0, 1, \dots, 7$$

which can be also checked by direct calculations such as

$$\begin{aligned} (\alpha e_4)(\alpha e_7) &= a_4 a_7 = a_4(a_3 a_4) = -a_4(a_4 a_3) = a_3 = \alpha e_3 = \alpha(e_4 e_7), \\ (\alpha e_1)(\alpha e_6) &= a_1 a_6 = a_1(a_4 a_2) = -a_4(a_1 a_2) = -a_4 a_3 = a_3 a_4 = a_7 \\ &= \alpha e_7 = \alpha(e_1 e_6), \\ (\alpha e_3)(\alpha e_6) &= a_3 a_6 = (a_1 a_2)(a_4 a_2) = -(a_2 a_1)(a_4 a_2) = -a_2(a_1 a_4) a_2 \\ &= -a_2 a_5 a_2 = a_2 a_2 a_5 = -a_5 = -\alpha e_5 = \alpha(e_3 e_6), \text{ etc.} \end{aligned}$$

Hence $\alpha \in G_2$ and $\alpha e_1 = a_1$, and so $\alpha^{-1}a_1 = e_1$. This shows the transitivity. The isotropy subgroup $(G_2)_{e_1}$ of G_2 at e_1 is $SU(3)$ (Theorem 1.9.1). Thus we have the homeomorphism $G_2/SU(3) \simeq S^6$.

Since S^6 and $SU(3)$ are both simply connected, from $G_2/SU(3) \simeq S^6$ (Theorem 1.9.2), we see that G_2 is also simply connected. Hence we have the following theorem.

Theorem 1.9.3. $G_2 = \{\alpha \in \text{Iso}_{\mathbf{R}}(\mathfrak{C}) \mid \alpha(xy) = (\alpha x)(\alpha y)\}$ is a simply connected compact simple Lie group.

Remark 1. Since G_2 is connected, G_2 is contained in $SO(7) = \{\alpha \in O(7) \mid \det \alpha = 1\}$: $G_2 \subset SO(7)$.

Remark 2. Since we know that the dimension of the group G_2 as $\dim G_2 = \dim \mathfrak{g}_2 = 14$ (Theorem 1.4.3), $G_2/SU(3) \simeq S^6$ is proved as follows. The group G_2 acts on S^6 . The isotropy subgroup $(G_2)_{e_1}$ of G_2 at e_1 is $SU(3)$ (Theorem 1.9.1) and $\dim(G_2/(G_2)_{e_1}) = \dim G_2 - \dim SU(3) = 14 - 8 = 6 = \dim S^6$. Therefore we have $G_2/SU(3) \simeq S^6$.

Using the mapping $\varphi : SU(3) \rightarrow G_2$, we define a mapping $w : \mathfrak{C} \rightarrow \mathfrak{C}$ by

$$w = \varphi(\text{diag}(\omega_1, \omega_1, \omega_1))$$

where $\omega_1 = -\frac{1}{2} + \frac{\sqrt{3}}{2}e_1 \in \mathbf{C} \subset \mathfrak{C}$. This w is defined as

$$w(a + \mathbf{m}) = a + \omega_1 \mathbf{m}, \quad a + \mathbf{m} \in \mathbf{C} \oplus \mathbf{C}^3 = \mathfrak{C}.$$

Then $w \in G_2$ and $w^3 = 1$.

We shall study the following subgroup $(G_2)^w$ of G_2 :

$$(G_2)^w = \{\alpha \in G_2 \mid w\alpha = \alpha w\}.$$

Theorem 1.9.4. $(G_2)^w = (G_2)_{e_1} \cong SU(3)$.

Proof. Recall the mapping $\varphi : SU(3) \rightarrow G_2$ of Theorem 1.9.1. We first show that $\varphi(SU(3)) \in (G_2)^w$. Indeed, for $A \in SU(3)$ and $a + \mathbf{m} \in \mathbf{C} \oplus \mathbf{C}^3 = \mathfrak{C}$, we have

$$\begin{aligned} w\varphi(A)(a + \mathbf{m}) &= w(a + A\mathbf{m}) = a + \omega_1 A\mathbf{m} \\ &= a + A\omega_1 \mathbf{m} = \varphi(A)w(a + \mathbf{m}). \end{aligned}$$

Hence $w\varphi(A) = \varphi(A)w$, so that $\varphi(A) \in (G_2)^w$. Conversely, let $\alpha \in (G_2)^w$. We consider the \mathbf{R} -vector subspace $\mathfrak{C}_w = \{x \in \mathfrak{C} \mid wx = x\}$ of \mathfrak{C} , then $\mathfrak{C}_w = \mathbf{C}$. Since α satisfies $w\alpha = \alpha w$, \mathfrak{C}_w is invariant under α . Since the restriction of α to \mathfrak{C}_w induces an automorphism of \mathbf{C} , we have

$$\alpha e_1 = e_1 \quad \text{or} \quad \alpha e_1 = -e_1.$$

In the latter case, consider the mapping $\gamma_1 : \mathfrak{C} \rightarrow \mathfrak{C}$ defined by $\gamma_1(a + \mathbf{m}) = \bar{a} + \overline{\mathbf{m}}$. Then we have $\gamma_1 \in G_2$ and $\gamma_1 e_1 = -e_1$. Let $\beta = \gamma_1 \alpha$. Since $\beta e_1 = e_1$, we have $\beta \in SU(3) \subset (G_2)_{e_1}$ (Theorem 1.9.1) $\subset (G_2)^w$. Therefore, $\gamma_1 = \beta \alpha^{-1} \in (G_2)^w$, which is a contradiction. Indeed,

$$\omega_1 \overline{\mathbf{m}} = w(\gamma_1 \mathbf{m}) = \gamma_1(w\mathbf{m}) = \overline{\omega_1 \mathbf{m}} = \overline{\omega_1} \overline{\mathbf{m}} \quad \text{for all } \mathbf{m} \in \mathbf{C}^3,$$

which is false. Therefore $\alpha e_1 = e_1$, and so $\alpha \in SU(3)$ (Theorem 1.9.1). Thus we have $(G_2)^w = (G_2)_{e_1}$.

1.10. Involution γ and subgroup $(Sp(1) \times Sp(1))/\mathbf{Z}_2$ of G_2

The Cayley algebra \mathfrak{C} naturally contains the field \mathbf{H} of quaternions as

$$\mathbf{H} = \{x_0 + x_1 e_1 + x_2 e_2 + x_3 e_3 \mid x_i \in \mathbf{R}\}.$$

Any element $x \in \mathfrak{C}$ is expressed by

$$\begin{aligned} x &= x_0 + x_1 e_1 + x_2 e_2 + x_3 e_3 + x_4 e_4 + x_5 e_5 + x_6 e_6 + x_7 e_7 \quad (x_i \in \mathbf{R}) \\ &= (x_0 + x_1 e_1 + x_2 e_2 + x_3 e_3) + (x_4 + x_5 e_1 - x_6 e_2 + x_7 e_3) e_4, \end{aligned}$$

that is,

$$x = m + a e_4, \quad m, a \in \mathbf{H}.$$

In $\mathbf{H} \oplus \mathbf{H} e_4$, we define a multiplication, an inner product $(\ , \)$, a conjugation $\overline{}$ and an \mathbf{R} -linear transformation γ respectively by

$$\begin{aligned} (m + a e_4)(n + b e_4) &= (mn - \bar{b}a) + (a\bar{n} + bm)e_4, \\ (m + a e_4, n + b e_4) &= (m, n) + (a, b), \\ \overline{m + a e_4} &= \bar{m} - a e_4, \\ \gamma(m + a e_4) &= m - a e_4. \end{aligned}$$

Since these operations correspond to their respective operations in \mathfrak{C} , hereafter, we identify $\mathbf{H} \oplus \mathbf{H} e_4$ with \mathfrak{C} , that is,

$$\mathbf{H} \oplus \mathbf{H} e_4 = \mathfrak{C}.$$

We shall study the following subgroup $(G_2)^\gamma$ of G_2 :

$$(G_2)^\gamma = \{\alpha \in G_2 \mid \gamma \alpha = \alpha \gamma\}.$$

Theorem 1.10.1. $(G_2)^\gamma \cong (Sp(1) \times Sp(1))/\mathbf{Z}_2$, $\mathbf{Z}_2 = \{(1, 1), (-1, -1)\}$.

Proof. We define a mapping $\varphi : Sp(1) \times Sp(1) \rightarrow (G_2)^\gamma$ by

$$\varphi(p, q)(m + a e_4) = qm\bar{q} + (pa\bar{q})e_4, \quad m + a e_4 \in \mathbf{H} \oplus \mathbf{H} e_4 = \mathfrak{C}.$$

We first show that $\varphi(p, q) \in (G_2)^\gamma$. For $\alpha = \varphi(p, q)$, $p, q \in Sp(1)$ and $x = m + ae_4$, $y = n + be_4 \in \mathbf{H} \oplus \mathbf{H}e_4 = \mathfrak{C}$, we have

$$\begin{aligned} (\alpha x)(\alpha y) &= (qm\bar{q} + (pa\bar{q})e_4)(qn\bar{q} + (pb\bar{q})e_4) \\ &= ((qm\bar{q})(qn\bar{q}) - (\overline{pb\bar{q}})(pa\bar{q})) + ((pa\bar{q})(\overline{qn\bar{q}}) + (pb\bar{q})(qm\bar{q}))e_4 \\ &= q(mn - \bar{b}a)\bar{q} + (p(a\bar{n} + bm)\bar{q})e_4 \\ &= \varphi(p, q)((m + ae_4)(n + be_4)) = \alpha(xy). \end{aligned}$$

Hence $\varphi(p, q) \in G_2$. Clearly $\gamma\varphi(p, q) = \varphi(p, q)\gamma$, so that $\varphi(p, q) \in (G_2)^\gamma$. Evidently φ is a homomorphism. We shall show that φ is onto. Let $\alpha \in (G_2)^\gamma$. Since α satisfies $\gamma\alpha = \alpha\gamma$, $\mathfrak{C}_\gamma = \{x \in \mathfrak{C} \mid \gamma x = x\} = \mathbf{H}$ is invariant under α , so that the restriction of α to \mathfrak{C}_γ induces an automorphism of \mathbf{H} . Hence there exists $q \in Sp(1)$ satisfying

$$\alpha m = qm\bar{q}, \quad m \in \mathbf{H}$$

(Proposition 108 of Yokota [58]). By putting $\beta = \varphi(1, q)^{-1}\alpha$, we have $\beta \in (G_2)^\gamma$ and $\beta|_{\mathbf{H}} = 1$. Therefore β induces a transformation of $\mathfrak{C}_{-\gamma} = \{x \in \mathfrak{C} \mid \gamma x = -x\} = \mathbf{H}e_4$. By putting $\beta e_4 = pe_4$, where $p \in \mathbf{H}$, we have $|p| = |pe_4| = |\beta e_4| = |e_4| = 1$, which implies $p \in Sp(1)$. From

$$\beta(m + ae_4) = \beta m + (\beta a)(\beta e_4) = m + a(pe_4) = m + (pa)e_4 = \varphi(p, 1)(m + ae_4),$$

we have $\beta = \varphi(p, 1)$. Therefore we have

$$\alpha = \varphi(1, q)\beta = \varphi(1, q)\varphi(p, 1) = \varphi(p, q), \quad (p, q) \in Sp(1) \times Sp(1),$$

which shows that φ is onto. $\text{Ker } \varphi = \{(1, 1), (-1, -1)\} = \mathbf{Z}_2$ is easily obtained. Thus we have the isomorphism $(Sp(1) \times Sp(1))/\mathbf{Z}_2 \cong (G_2)^\gamma$.

Remark 1. $(Sp(1) \times Sp(1))/\mathbf{Z}_2 \cong SO(4)$.

Indeed, a mapping $f : Sp(1) \times Sp(1) \rightarrow SO(4) = SO(\mathbf{H})$ defined by

$$f(p, q)x = px\bar{q}, \quad x \in \mathbf{H}$$

induces the isomorphism $(Sp(1) \times Sp(1))/\mathbf{Z}_2 \cong SO(4)$ as groups.

Remark 2. Since $(G_2)^\gamma$ is connected as the fixed points subgroup by an automorphism γ of the simply connected group G_2 , the fact that $\varphi : Sp(1) \times Sp(1) \rightarrow (G_2)^\gamma$ is onto can be proved as follows. The elements

$$\begin{aligned} 2G_{12} - G_{47} - G_{56}, & \quad -G_{47} + G_{56}, \\ 2G_{13} - G_{46} - G_{57}, & \quad G_{46} + G_{57}, \\ 2G_{23} - G_{45} - G_{67}, & \quad -G_{45} + G_{67} \end{aligned}$$

forms an \mathbf{R} -basis of $(\mathfrak{g}_2)^\gamma$. So $\dim(\mathfrak{g}_2)^\gamma = 6 = 3 + 3 = \dim(\mathfrak{sp}(1) \oplus \mathfrak{sp}(1))$. Hence φ is onto.

1.11. Center $z(G_2)$ of G_2

Theorem 1.11.1. *The center $z(G_2)$ of the group G_2 is trivial:*

$$z(G_2) = \{1\}.$$

Proof. Let $\alpha \in z(G_2)$. From the commutativity with γ : $\gamma\alpha = \alpha\gamma$, we have $\alpha \in SO(4)$ (Remark 1 of Theorem 1.10.1) and so $\alpha \in z(SO(4))$. Since the center $z(SO(4))$ of $SO(4)$ is the group of order 2, we have

$$\alpha = 1 \quad \text{or} \quad \alpha = \gamma.$$

However $\gamma \notin z(G_2)$ (Theorem 1.10.1). Hence $\alpha = 1$.

1.12. Complex exceptional Lie group G_2^C

Definition. The group G_2^C is defined to be the automorphism group of the complex Cayley algebra \mathfrak{C}^C :

$$G_2^C = \{\alpha \in \text{Iso}_C(\mathfrak{C}^C) \mid \alpha(xy) = (\alpha x)(\alpha y)\}.$$

Lemma 1.12.1. *For $\alpha \in G_2^C$, we have*

$$(\alpha x, \alpha y) = (x, y), \quad x, y \in \mathfrak{C}^C.$$

Proof. The equality $\alpha x = (\alpha 1)(\alpha x)$ holds for all $x \in \mathfrak{C}^C$, and so we have $\alpha 1 = 1$. We shall show that

$$\overline{\alpha e_k} = -\alpha e_k, \quad k = 1, 2, \dots, 7.$$

Note that $(\alpha e_k)(\alpha e_k) = \alpha(e_k e_k) = \alpha(-1) = -1$. Let $x = \alpha e_k$ and $N(x) = x\bar{x} \in C$. Then $xx = -1$ and $N(x)N(x) = 1$, which shows that $N(x) = \pm 1$. If $N(x) = -1$, then $\bar{x} = -xx\bar{x} = -xN(x) = x$, so $x \in C$. From $xx = -1$, we have $x = \pm i$. Then $\alpha(e_k) = \pm\alpha(i)$, and so $e_k = \pm i$, which is a contradiction. Hence $N(x) = 1$, that is, $x\bar{x} = 1$ and $\bar{x} = -xx\bar{x} = -xN(x) = -x$. Thus we have

$$\overline{\alpha x} = \alpha \bar{x}, \quad x \in \mathfrak{C}^C.$$

Now we have

$$\begin{aligned} (\alpha x, \alpha y) &= \frac{1}{2}((\alpha x)(\overline{\alpha y}) + (\alpha y)(\overline{\alpha x})) = \frac{1}{2}((\alpha x)(\alpha \bar{y}) + (\alpha y)(\alpha \bar{x})) \\ &= \alpha\left(\frac{1}{2}(\alpha(x\bar{y} + y\bar{x}))\right) = \alpha((x, y)) = (x, y). \end{aligned}$$

We define a positive definite Hermitian inner product $\langle x, y \rangle$ in \mathfrak{C}^C by

$$\langle x, y \rangle = (\tau x, y),$$

For $\alpha \in \text{Hom}_C(\mathfrak{C}^C)$, we denote the complex conjugate transpose of α with respect to the inner product $\langle x, y \rangle$ by α^* : $\langle \alpha^* x, y \rangle = \langle x, \alpha y \rangle$.

Lemma 1.12.2. (1) For $\alpha \in G_2^C$, we have $\alpha^* = \tau \alpha^{-1} \tau \in G_2^C$.

(2) For any $\alpha \in G_2$, its complexified mapping $\alpha^C : \mathfrak{C}^C \rightarrow \mathfrak{C}^C$ belongs to G_2^C : $\alpha^C \in G_2^C$. Identifying α with α^C , we regard G_2 as a subgroup of G_2^C : $G_2 \subset G_2^C$. Now, for $\alpha \in G_2^C$, we have, $\alpha \in G_2$ if and only if $\tau \alpha = \alpha \tau$, that is,

$$G_2 = \{\alpha \in G_2^C \mid \tau \alpha = \alpha \tau\}.$$

Proof. (1) $\langle \alpha^* x, y \rangle = \langle x, \alpha y \rangle = \langle \tau x, \alpha y \rangle = \langle \alpha^{-1} \tau x, y \rangle = \langle \tau \alpha^{-1} \tau x, y \rangle$ for all $x, y \in \mathfrak{C}^C$. Hence $\alpha^* = \tau \alpha^{-1} \tau \in G_2^C$.

(2) Let $\alpha \in G_2^C$ satisfy $\tau \alpha = \alpha \tau$. Then since $\tau \alpha x = \alpha \tau x = \alpha x$, we have $\alpha x \in \mathfrak{C}$ for $x \in \mathfrak{C}$. Hence α induces an \mathbf{R} -transformation α' of \mathfrak{C} and $\alpha' \in G_2$, further we have $\alpha = (\alpha')^C$.

Theorem 1.12.3. The polar decomposition of the group G_2^C is given by

$$G_2^C \simeq G_2 \times \mathbf{R}^{14}.$$

In particular, G_2^C is a simply connected complex Lie group of type G_2 .

Proof. Evidently G_2^C is an algebraic subgroup of $\text{Iso}_C(\mathfrak{C}^C) = GL(8, C)$. If $\alpha \in G_2^C$, then $\alpha^* \in G_2^C$ (Lemma 1.12.2.(1)). Hence, from Chevalley's lemma (Chevalley [5]), we have

$$G_2^C \simeq (G_2^C \cap U(\mathfrak{C}^C)) \times \mathbf{R}^d = G_2 \times \mathbf{R}^d,$$

where $U(\mathfrak{C}^C) = \{\alpha \in \text{Iso}_C(\mathfrak{C}^C) \mid \langle \alpha X, \alpha Y \rangle = \langle X, Y \rangle\}$ and $d = \dim G_2^C - \dim G_2 = 2 \times 14 - 14 = 14$. Since G_2 is simply connected (Theorem 1.9.3), G_2^C is also simply connected. The Lie algebra of the group G_2^C is \mathfrak{g}_2^C , so that G_2^C is a complex simple Lie group of type G_2 .

1.13. Non-compact exceptional Lie group $G_{2(2)}$ of type G_2

In $\mathfrak{C}' = \mathbf{H} \oplus \mathbf{H}e_4'$, we define a multiplication by

$$(m + ae_4')(n + be_4') = (mn + \bar{b}a) + (a\bar{n} + bm)e_4'.$$

This algebra \mathfrak{C}' is called the split Cayley algebra, and is isomorphic to $(\mathfrak{C}^C)_{\tau\gamma} = \{x \in \mathfrak{C}^C \mid \tau\gamma x = x\}$ as algebras. Now, the group $G_{2(2)}$ is defined to be the automorphism group of the split Cayley algebras \mathfrak{C}' :

$$G_{2(2)} = \{\alpha \in \text{Iso}_{\mathbf{R}}(\mathfrak{C}') \mid \alpha(xy) = (\alpha x)(\alpha y)\}.$$

and which can also be defined by

$$G_{2(2)} = (G_2^C)^{\tau\gamma} = \{\alpha \in G_2^C \mid \tau\gamma\alpha = \alpha\gamma\tau\}.$$

Theorem 1.13.1. *The polar decomposition of the Lie groups $G_{2(2)}$ is given by*

$$G_{2(2)} \simeq (Sp(1) \times Sp(1))/\mathbf{Z}_2 \times \mathbf{R}^8.$$

Proof. In the split Cayley algebra \mathfrak{C}' , the inner product $(x, y)'$ is defined as $(x, y)' = \frac{1}{2}(x\bar{y} + y\bar{x})$. If we define an inner product (x, y) of \mathfrak{C}' by $(\gamma x, y)'$, then (x, y) is a positive definite inner product. For $\alpha \in G_{2(2)}$, the transpose ${}^t\alpha$ of α with respect to the inner product (x, y) is ${}^t\alpha = \gamma\alpha^{-1}\gamma \in G_{2(2)}$. Since $G_{2(2)}$ is an algebraic subgroup of $\text{Iso}_{\mathbf{R}}(\mathfrak{C}') = GL(8, \mathbf{R})$, from Chevalley's lemma, we have

$$\begin{aligned} G_{2(2)} &\simeq (G_{2(2)} \cap O(8)) \times \mathbf{R}^d \\ &= ((G_2^C)^{\tau\gamma})^\gamma \times \mathbf{R}^d = ((G_2^C)^\tau)^\gamma \times \mathbf{R}^d = (G_2)^\gamma \times \mathbf{R}^d \\ &= (Sp(1) \times Sp(1))/\mathbf{Z}_2 \times \mathbf{R}^d \quad (\text{Theorem 1.10.1}), \quad d = 8. \end{aligned}$$

where $O(8) = \{\alpha \in \text{Iso}_{\mathbf{R}}(\mathfrak{C}') \mid (\alpha x, \alpha y) = (x, y)\}$.

Theorem 1.13.2. *The center $z(G_{2(2)})$ of the group $G_{2(2)}$ is trivial:*

$$z(G_{2(2)}) = \{1\}.$$

1.14. Principle of triality in $SO(8)$

For $a \in S^{n-1} = \{a \in \mathbf{R}^n \mid (a, a) = 1\}$, we define an element $D_a \in O(n) = O(\mathbf{R}^n) = \{A \in \text{Iso}_{\mathbf{R}}(\mathbf{R}^n) \mid (Ax, Ay) = (x, y)\}$ by

$$D_a x = x - 2(x, a)a, \quad x \in \mathbf{R}^n.$$

D_a is called the reflection with respect to the hyperplane orthogonal to a . Its determinant is -1 : $\det(D_a) = -1$.

Lemma 1.14.1. *The group $O(n)$ is generated by reflections, that is, any $A \in O(n)$ can be expressed by the product of finite number of reflections:*

$$A = D_{a_m} \cdots D_{a_2} D_{a_1}, \quad a_i \in S^{n-1}.$$

In particular, $A \in SO(n)$ can be expressed by the product of even number of reflections.

$$A = D_{a_{2m}} \cdots D_{a_2} D_{a_1}, \quad a_i \in S^{n-1}.$$

Proof. (See Theorem 24 of Yokota [58].)

From now on, the group $SO(8)$ is identified with the group

$$SO(\mathfrak{C}) = \{\alpha \in \text{Iso}_{\mathbf{R}}(\mathfrak{C}) \mid (\alpha x, \alpha y) = (x, y), \det \alpha = 1\}.$$

Theorem 1.14.2. (Principle of triality in $SO(8)$). For any $\alpha_3 \in SO(8)$, there exist $\alpha_1, \alpha_2 \in SO(8)$ such that

$$(\alpha_1 x)(\alpha_2 y) = \alpha_3(xy), \quad x, y \in \mathfrak{C}.$$

Moreover, α_1, α_2 are determined uniquely for α_3 up to the sign, that is, for α_3 , such α_1, α_2 have to be α_1, α_2 or $-\alpha_1, -\alpha_2$.

Proof. Since $\alpha_3 \in SO(8)$ is expressed by the product of even number of reflections (Lemma 1.14.1), it is sufficient to show the existence of α_1, α_2 only for $\alpha_3 = D_b D_a$, $a, b \in \mathfrak{C}, |a| = |b| = 1$. Now, since

$$D_a x = x - 2(x, a)a = x - (x\bar{a} + a\bar{x})a = -a\bar{x}a, \quad x \in \mathfrak{C},$$

we have $\alpha_3 x = D_b D_a x = b(\bar{a}x\bar{a})b$. Define mappings $\alpha_1, \alpha_2 : \mathfrak{C} \rightarrow \mathfrak{C}$ respectively by $\alpha_1 x = b(\bar{a}x)$, $\alpha_2 x = (x\bar{a})b$. We then see that $\alpha_1, \alpha_2 \in SO(8)$ and

$$(\alpha_1 x)(\alpha_2 y) = (b(\bar{a}x))((y\bar{a})b) = b(\bar{a}(xy)\bar{a})b = \alpha_3(xy), \quad x, y \in \mathfrak{C}.$$

Next, we shall show the uniqueness of α_1, α_2 up to the sign. To prove this, it is sufficient to show in the case $\alpha_3 = 1$. Now, let

$$(\alpha_1 x)(\alpha_2 y) = xy, \quad x, y \in \mathfrak{C}.$$

Let $\alpha_1 1 = p$, then $|p| = 1$ and $p(\alpha_2 y) = y$, so $\alpha_2 y = \bar{p}y$. Similarly we have $\alpha_1 x = x\bar{q}$, where $q = \alpha_2 1$. Hence $(x\bar{q})(\bar{p}y) = xy$. If we let $x = y = 1$, then $\bar{q}\bar{p} = 1$, so $\bar{q} = p$. Therefore we have

$$(xp)(\bar{p}y) = xy, \quad x, y \in \mathfrak{C}.$$

Putting py instead of y , we have

$$(xp)y = x(py), \quad x, y \in \mathfrak{C}.$$

From this, we see that p is a real number. Hence $p = \pm 1$ because $|p| = 1$. Thus we have $\alpha_1 = \alpha_2 = 1$ or $\alpha_1 = \alpha_2 = -1$.

Lemma 1.14.3. For $\alpha_1, \alpha_2, \alpha_3 \in O(8)$, the relation

$$(\alpha_1 x)(\alpha_2 y) = \overline{\alpha_3(\overline{xy})}, \quad x, y \in \mathfrak{C}$$

implies

$$\begin{aligned} (\alpha_2 x)(\alpha_3 y) &= \overline{\alpha_1(\overline{xy})}, \quad x, y \in \mathfrak{C}, \\ (\alpha_3 x)(\alpha_1 y) &= \overline{\alpha_2(\overline{xy})}, \quad x, y \in \mathfrak{C}. \end{aligned}$$

Proof. If $x = 0$ or $y = 0$, then the statement is trivially valid, so we may assume $x, y \neq 0$. Now, multiply $\overline{\alpha_1 x}$ from left and $\alpha_3(\overline{xy})$ from right on the relation

$(\alpha_1 x)(\alpha_2 y) = \overline{\alpha_3(\overline{xy})}$. Then we get $|x|^2(\alpha_2 y)(\alpha_3(\overline{xy})) = \overline{\alpha_1 x}|xy|^2$. Therefore

$$(\alpha_2 y)(\alpha_3(\overline{xy})) = \overline{\alpha_1 x}|y|^2.$$

Putting \overline{yz} instead of x , we have $(\alpha_2 y)(\alpha_3(\overline{yyz})) = \overline{\alpha_1(\overline{yz})}|y|^2$, that is,

$$(\alpha_2 y)(\alpha_3 z) = \overline{\alpha_1(\overline{yz})}.$$

The other relation is similarly obtained.

Lemma 1.14.4. *If $\alpha_1, \alpha_2, \alpha_3 \in O(8)$ satisfy*

$$(\alpha_1 x)(\alpha_2 y) = \alpha_3(xy), \quad x, y \in \mathfrak{C},$$

then $\alpha_1, \alpha_2, \alpha_3 \in SO(8)$.

Proof. Suppose $\alpha_1 \notin SO(8)$. Since the mapping $\epsilon : \mathfrak{C} \rightarrow \mathfrak{C}$ defined by $\epsilon x = \overline{\alpha_1 x}$ belongs to $O(8)$, and since $\det \epsilon = -1$, we have $\beta_1 = \epsilon \alpha_1^{-1} \in SO(8)$. Using the principle of triality (Theorem 1.14.2) on the element β_1 (cf. Lemma 1.14.3), there exist $\beta_2, \beta_3 \in SO(8)$ such that

$$(\beta_1(\alpha_1 x))(\beta_2(\alpha_2 y)) = \beta_3((\alpha_1 x)(\alpha_2 y)) = \beta_3(\alpha_3(xy)), \quad x, y \in \mathfrak{C}.$$

Setting $\beta_2 \alpha_2 = \gamma_2$ and $\beta_3 \alpha_3 = \gamma_3$, the above relation becomes

$$\overline{\alpha_1}(\gamma_2 y) = \gamma_3(xy), \quad x, y \in \mathfrak{C}.$$

Put $x = 1$, then $\gamma_2 y = \gamma_3 y$, so $\gamma_2 = \gamma_3$. Hence we have

$$\overline{\alpha_1}(\gamma_2 y) = \gamma_2(xy), \quad x, y \in \mathfrak{C}. \tag{i}$$

Put $\gamma_2 1 = p$, then $|p| = 1$. Let $y = 1$ in (i), then $\overline{\alpha_1} p = \gamma_2 x$. Hence (i) becomes

$$\overline{\alpha_1}(\overline{y}p) = (\overline{xy})p, \quad x, y \in \mathfrak{C}. \tag{ii}$$

Again let $y = p$, then $\overline{\alpha_1} = (\overline{xp})p$, and so $\overline{\alpha_1} \overline{p} = \overline{xp}$. Then

$$px = xp \quad \text{for all } x \in \mathfrak{C}.$$

Therefore $p \in \mathbf{R}$, hence $p = \pm 1$ since $|p| = 1$. Thus (ii) becomes $\overline{\alpha_1} \overline{y} = \overline{xy}$, and so

$$xy = yx \quad \text{for all } x, y \in \mathfrak{C}.$$

But this is a contradiction. Therefore $\alpha_1 \in SO(8)$. As for α_2, α_3 , use Lemma 1.14.3 and the argument above, to show $\alpha_2, \alpha_3 \in SO(8)$.

As a corollary of the principle of triality (Theorem 1.14.2), we have the following proposition.

Proposition 1.14.5. *For $a \in \mathfrak{C}$ such that $|a| = 1$, the mapping $\alpha_a : \mathfrak{C} \rightarrow \mathfrak{C}$ defined by*

$$\alpha_a x = axa^{-1}, \quad x \in \mathfrak{C}$$

belongs to the group G_2 if and only if $a^3 = \pm 1$.

Proof. If we apply Lemma 1.14.3 to the Moufang formula $(\bar{a}x)(y\bar{a}) = \bar{a}(xy)\bar{a}$, then

$$(x\bar{a})(aya) = (xy)a, \quad x, y \in \mathfrak{C}.$$

If we replace x by ax and y by ya respectively, then

$$(ax\bar{a})(aya^2) = a(xy)a^2, \quad x, y \in \mathfrak{C}.$$

Now, the mapping α_a belongs to G_2 if and only if

$$(ax\bar{a})(ay\bar{a}) = a(xy)\bar{a}, \quad x, y \in \mathfrak{C}.$$

From the uniqueness of the principle of triality up to sign, we have

$$aya^2 = \pm ay\bar{a}.$$

Therefore $a^2 = \pm \bar{a}$, so that $a^3 = \pm 1$. The converse also holds.

Let $\omega_1 = -\frac{1}{2} + \frac{\sqrt{3}}{2}e_1 \in \mathbf{C} \subset \mathfrak{C}$. Then $\omega_1^3 = 1$, so $\alpha_{\bar{\omega}_1} \in G_2$ by Proposition 1.14.5. This $\alpha_{\bar{\omega}_1}$ is nothing but w of Section 1.9: $\alpha_{\bar{\omega}_1} = w$, because

$$\alpha_{\bar{\omega}_1}(a + \mathbf{m}) = \bar{\omega}_1(a + \mathbf{m})\omega_1 = \bar{\omega}_1 a \omega_1 + \bar{\omega}_1^2 \mathbf{m} = a + \omega_1 \mathbf{m} = w(a + \mathbf{m}),$$

for $a + \mathbf{m} \in \mathbf{C} \oplus \mathbf{C}^3 = \mathfrak{C}$.

1.15. Spinor group $Spin(7)$

If we use the principle of triality for $\alpha \in SO(7)$, then there exist $\tilde{\alpha}, \alpha' \in SO(8)$ satisfying

$$(\alpha x)(\tilde{\alpha} y) = \alpha'(xy), \quad x, y \in \mathfrak{C}. \tag{i}$$

Putting $x = 1$, we have $\tilde{\alpha} y = \alpha' y$, and so $\tilde{\alpha} = \alpha'$. Then (i) becomes

$$(\alpha x)(\tilde{\alpha} y) = \tilde{\alpha}(xy), \quad x, y \in \mathfrak{C}. \tag{ii}$$

Conversely, suppose that $\alpha, \tilde{\alpha} \in SO(8)$ satisfy (ii). Putting $x = 1$, we have $(\alpha 1)(\tilde{\alpha} y) = \tilde{\alpha} y$, and so we have $\alpha 1 = 1$. Hence $\alpha \in SO(7)$.

Definition. We define a subgroup \tilde{B}_3 of $SO(8)$ by

$$\tilde{B}_3 = \{\tilde{\alpha} \in SO(8) \mid (\alpha x)(\tilde{\alpha} y) = \tilde{\alpha}(xy), x, y \in \mathfrak{C} \text{ for some } \alpha \in SO(7)\}.$$

\tilde{B}_3 is a compact group.

Theorem 1.15.1. $\tilde{B}_3/G_2 \simeq S^7$, $\tilde{B}_3 \cap SO(7) = G_2$.

In particular, the group \tilde{B}_3 is connected.

Proof. $S^7 = \{a \in \mathfrak{C} \mid |a| = 1\}$ is a 7 dimensional sphere. Since \tilde{B}_3 is a subgroup of $SO(8)$, the group \tilde{B}_3 acts on S^7 . We shall show that \tilde{B}_3 acts transitively on S^7 . To prove this, it is sufficient to show that any $b_0 \in S^7$ can be transformed to $1 \in S^7$ by some $\alpha \in \tilde{B}_3$. Now, we choose any element $a_1 \in S^7$ such that $(1, a_1) = 0$. and choose any element $a_2 \in S^7$ such that $(1, a_2) = (a_1, a_2) = 0$. Let $a_3 \in S^7$ be the element determined by $a_3 b_0 = a_2(a_1 b_0)$. More precisely, if we let $a_3 = (a_2(a_1 b_0))\bar{b}_0$, then $a_3 \in S^7$ and satisfies $(1, a_3) = (a_1, a_3) = (a_2, a_3) = 0$. Choose any element $a_4 \in S^7$ such that $(1, a_4) = (a_1, a_4) = (a_2, a_4) = (a_3, a_4) = 0$. Let $a_5, a_6, a_7 \in S^7$ be elements determined by

$$a_5 b_0 = a_1(a_4 b_0), \quad a_6 b_0 = a_2(a_4 b_0), \quad a_7 b_0 = a_6(a_1 b_0).$$

Then, $\{a_0 = 1, a_1, a_2, \dots, a_7\}$ forms an orthonormal \mathbf{R} -basis of \mathfrak{C} . To prove this, we need to show $(a_i, a_j) = \delta_{ij}$, $i, j = 0, 1, \dots, 7$. We will only show the following two since the others can be proved in a similar manner.

$$\begin{aligned} (a_2, a_6) &= (a_2 b_0, a_6 b_0) = (a_2 b_0, a_2(a_4 b_0)) = (b_0, a_4 b_0) = (1, a_4) = 0, \\ (a_3, a_7) &= (a_3 b_0, a_7 b_0) = (a_2(a_1 b_0), a_6(a_1 b_0)) = (a_1(a_2 b_0), a_1(a_6 b_0)) \\ &= (a_2 b_0, a_6 b_0) = (a_2 b_0, a_2(a_4 b_0)) = (b_0, a_4 b_0) = (1, a_4) = 0. \end{aligned}$$

Now, since $\{e_0, e_1, \dots, e_7\}$ and $\{a_0 = 1, a_1, \dots, a_7\}$ are both orthonormal \mathbf{R} -bases of \mathfrak{C} , the \mathbf{R} -linear isomorphism $\alpha : \mathfrak{C} \rightarrow \mathfrak{C}$ satisfying

$$\alpha e_i = a_i, \quad i = 0, 1, \dots, 7$$

belongs to $O(7)$. Moreover, this α satisfies

$$(\alpha x)((\alpha y)b_0) = (\alpha(xy))b_0, \quad x, y \in \mathfrak{C}. \quad (i)$$

To prove this, it is sufficient to show that

$$(\alpha e_i)((\alpha e_j)b_0) = (\alpha(e_i e_j))b_0, \quad i, j = 0, 1, \dots, 7.$$

Again we need to verify many cases, but here we will only show the following two examples.

$$\begin{aligned} (\alpha e_1)((\alpha e_3)b_0) &= a_1(a_3 b_0) = a_1(a_2(a_1 b_0)) = -a_1(a_1(a_2 b_0)) = a_2 b_0 \\ &= \alpha(e_2)b_0 = \alpha(e_1 e_3)b_0, \\ (\alpha e_2)((\alpha e_5)b_0) &= a_2(a_5 b_0) = a_2(a_1(a_4 b_0)) = -a_1(a_2(a_4 b_0)) = -a_1(a_6 b_0) \\ &= a_6(a_1 b_0) = a_7 b_0 = \alpha(e_7)b_0 = \alpha(e_2 e_5)b_0. \end{aligned}$$

Now, if we put

$$b_i = a_i b_0, \quad i = 0, 1, \dots, 7,$$

then $\{b_0, b_1, \dots, b_7\}$ is an orthonormal \mathbf{R} -basis of \mathfrak{C} . The \mathbf{R} -linear isomorphism $\tilde{\alpha} : \mathfrak{C} \rightarrow \mathfrak{C}$ satisfying

$$\tilde{\alpha} e_i = b_i, \quad i = 0, 1, \dots, 7$$

belongs to $O(8)$. Since $\tilde{\alpha} x = (\alpha x) b_0$, it follows from (i) that

$$(\alpha x)(\tilde{\alpha} y) = \tilde{\alpha}(xy), \quad x, y \in \mathfrak{C}. \quad (\text{ii})$$

Since $\alpha \in O(7)$, $\tilde{\alpha} \in O(8)$ satisfy (ii), $\alpha \in SO(7)$, $\tilde{\alpha} \in SO(8)$ (Lemma 1.14.4). Hence $\tilde{\alpha} \in \tilde{B}_3$ and $\tilde{\alpha} 1 = b_0$, and so $\tilde{\alpha}^{-1} b_0 = 1$. This shows the transitivity. The isotropy subgroup of \tilde{B}_3 at $1 \in S^7$ is G_2 . Indeed, if $\tilde{\alpha} \in \tilde{B}_3$ satisfies $\tilde{\alpha} 1 = 1$, then we have $\alpha = \tilde{\alpha}$, so $\tilde{\alpha} \in G_2$. Conversely, $\alpha \in G_2$ satisfies $\alpha \in \tilde{B}_3$ and $\alpha 1 = 1$. Thus we have the homeomorphism $\tilde{B}_3/G_2 \simeq S^7$.

Theorem 1.15.2. $\tilde{B}_3 \cong Spin(7)$.

(From now on, we identify these groups).

Proof. Suppose $\alpha \in SO(7)$ and $\tilde{\alpha} \in \tilde{B}_3$ satisfy the principle of triality

$$(\alpha x)(\tilde{\alpha} y) = \tilde{\alpha}(xy), \quad x, y \in \mathfrak{C}.$$

We define a mapping $p : \tilde{B}_3 \rightarrow SO(7)$ by $p(\tilde{\alpha}) = \alpha$. It is not difficult to see that p is a homomorphism. The principle of triality implies that p is onto and $\text{Ker } p = \{1, -1\}$. Next, we shall prove that p is continuous. From Lemma 1.14.3, we have

$$\alpha(xy) = (\tilde{\alpha} x) \overline{(\tilde{\alpha} y)}, \quad x, y \in \mathfrak{C}.$$

Consider the matrices of α and $\tilde{\alpha}$ with respect to the \mathbf{R} -basis $\{e_0, e_1, \dots, e_7\}$. Then we can see that each component of matrix α is a polynomial of components of matrix $\tilde{\alpha}$ (for example, $\alpha e_1 = (\tilde{\alpha} e_2) \overline{(\tilde{\alpha} e_3)}$). Therefore p is continuous. Hence we have the isomorphism

$$\tilde{B}_3/\{1, -1\} \cong SO(7).$$

Therefore \tilde{B}_3 is isomorphic to $Spin(7)$ as the universal covering group of $SO(7)$.

1.16. Spinor group $Spin(8)$

Definition. We define a subgroup \tilde{D}_4 of $SO(8) \times SO(8) \times SO(8)$ by

$$\tilde{D}_4 = \{(\alpha_1, \alpha_2, \alpha_3) \in SO(8) \times SO(8) \times SO(8) \mid (\alpha_1 x)(\alpha_2 y) = \overline{\alpha_3(xy)}, x, y \in \mathfrak{C}\}.$$

\tilde{D}_4 is a compact group.

Since an element $(\alpha, \tilde{\alpha}, \kappa \tilde{\alpha})$ of \tilde{D}_4 satisfies $(\alpha x)(\tilde{\alpha} y) = \tilde{\alpha}(xy)$, $x, y \in \mathfrak{C}$, we see that \tilde{D}_4 contains $Spin(7)$ as a subgroup under the identification

$$Spin(7) \ni \tilde{\alpha} \longleftrightarrow (\alpha, \tilde{\alpha}, \kappa\tilde{\alpha}) \in \tilde{D}_4.$$

Proposition 1.16.1. $\tilde{D}_4/Spin(7) \simeq S^7$.

In particular, the group \tilde{D}_4 is connected.

Proof. $S^7 = \{a \in \mathfrak{C} \mid |a| = 1\}$ is a 7 dimensional sphere. We define an action of \tilde{D}_4 on S^7 by

$$(\alpha_1, \alpha_2, \alpha_3)a = \alpha_1 a, \quad a \in S^7.$$

This action is transitive. Let $a \in S^7$. Since $SO(8)$ acts transitively on S^7 , there exists $\alpha_1 \in SO(8)$ such that $\alpha_1 1 = a$. For α_1 , choose $\alpha_2, \alpha_3 \in SO(8)$ satisfying the principle of triality

$$(\alpha_1 x)(\alpha_2 y) = \overline{\alpha_3(\overline{xy})}. \quad x, y \in \mathfrak{C}.$$

Then $(\alpha_1, \alpha_2, \alpha_3) \in \tilde{D}_4$ and $(\alpha_1, \alpha_2, \alpha_3)1 = a$, which shows the transitivity. The isotropy subgroup of \tilde{D}_4 at $1 \in S^7$ is $Spin(7)$. Indeed, if $(\alpha_1, \alpha_2, \alpha_3) \in \tilde{D}_4$ satisfies $(\alpha_1, \alpha_2, \alpha_3)1 = 1$, then $\alpha_1 1 = 1$. Therefore $\alpha_1 \in SO(7)$, which shows that $(\alpha_1, \alpha_2, \alpha_3) \in Spin(7)$ and vice versa. Thus we have the homeomorphism $\tilde{D}_4/Spin(7) \simeq S^7$.

Theorem 1.16.2. $\tilde{D}_4 \cong Spin(8)$.

(From now on, we identify these groups).

Proof. We define a mapping $p : \tilde{D}_4 \rightarrow SO(8)$ by

$$p(\alpha_1, \alpha_2, \alpha_3) = \alpha_1.$$

Evidently, p is a homomorphism. The principle of triality implies that p is onto and $\text{Ker } p = \{(1, 1, 1), (1, -1, -1)\}$. Thus we obtain the isomorphism

$$\tilde{D}_4/\{(1, 1, 1), (1, -1, -1)\} \cong SO(8).$$

Therefore \tilde{D}_4 is isomorphic to $Spin(8)$ as the universal covering group of $SO(8)$.

Theorem 1.16.3. The center $z(Spin(8))$ of the group $Spin(8)$ is isomorphic to the group $\mathbf{Z}_2 \times \mathbf{Z}_2$:

$$\begin{aligned} z(Spin(8)) &= \{(1, 1, 1), (1, -1, -1), (-1, -1, 1), (-1, 1, -1)\} \\ &= \{(1, 1, 1), (1, -1, -1)\} \times \{(-1, -1, 1), (-1, 1, -1)\} \cong \mathbf{Z}_2 \times \mathbf{Z}_2. \end{aligned}$$

Proof. The proof follows easily from $z(SO(8)) = \{1, -1\}$ and the principle of triality.

Theorem 1.16.4. We define automorphisms $\kappa, \pi, \nu : Spin(8) \rightarrow Spin(8)$ respectively by

$$\begin{aligned}\kappa(\alpha_1, \alpha_2, \alpha_3) &= (\kappa\alpha_1, \kappa\alpha_3, \kappa\alpha_2), \\ \pi(\alpha_1, \alpha_2, \alpha_3) &= (\kappa\alpha_3, \kappa\alpha_2, \kappa\alpha_1), \\ \nu(\alpha_1, \alpha_2, \alpha_3) &= (\alpha_2, \alpha_3, \alpha_1),\end{aligned}$$

where $\kappa : SO(8) \rightarrow SO(8)$ in the right side is defined by $(\kappa\alpha)x = \overline{\alpha x}$, $x \in \mathfrak{C}$. Then we have relations

$$\kappa^2 = 1, \quad \pi^2 = 1, \quad \nu^3 = 1, \quad \nu = \pi\kappa.$$

The subgroup \mathfrak{S}_3 generated by κ, π in the automorphism group $\text{Aut}(Spin(8))$ of $Spin(8)$ is isomorphic to the symmetric group S_3 . Also, we have

$$\begin{aligned}Spin(7) &= \{\alpha \in Spin(8) \mid \kappa\alpha = \alpha\}, \\ G_2 &= \{\alpha \in Spin(8) \mid \lambda\alpha = \alpha, \lambda \in \mathfrak{S}_3\} \\ &= \{\alpha \in Spin(8) \mid \pi\alpha = \alpha, \nu\alpha = \alpha\} \\ &= \{\alpha \in Spin(8) \mid \nu\alpha = \alpha\} \\ &= \{\alpha \in Spin(7) \mid \pi\alpha = \alpha\}.\end{aligned}$$

Moreover we have the isomorphism

$$Spin(8)/\{(1, 1, 1), (-1, -1, 1)\} \cong Spin(8)/\{(1, 1, 1), (1, -1, -1)\},$$

that is,

$$SO(8) \cong Ss(8).$$

Proof. The group multiplication between κ, π, ν is the same as that of Theorem 1.3.5. Next, we shall show

$$G_2 = \{\alpha \in Spin(8) \mid \nu\alpha = \alpha\}.$$

If $(\alpha_1, \alpha_2, \alpha_3) \in Spin(8)$ satisfies $\nu(\alpha_1, \alpha_2, \alpha_3) = (\alpha_1, \alpha_2, \alpha_3)$, then we have $\alpha_1 = \alpha_2 = \alpha_3 (= \alpha)$, that is,

$$(\alpha x)(\alpha y) = \kappa\alpha(xy), \quad x, y \in \mathfrak{C}, \tag{i}$$

Put $a = \alpha 1$, then $|a| = 1$, and put $x = 1$ and $y = 1$ in (i), then we have $a(\alpha y) = \kappa\alpha(y)$ and $(\alpha x)a = \kappa\alpha(x)$, respectively. Hence, we get

$$a(\alpha x) = (\alpha x)a \quad \text{for all } x \in \mathfrak{C}.$$

hence $a \in \mathbf{R}$, and so $a = \pm 1$ from $|a| = 1$. In the case $a = -1$, let $x = y = 1$ in (i), then $(-1)(-1) = -1$ which is a contradiction. Hence $a = 1$, so that $\kappa\alpha = \alpha$. Therefore we have $\alpha \in G_2$. Finally, the automorphism $\nu^2 : Spin(8) \rightarrow Spin(8)$ satisfies

$$\nu^2(-1, -1, 1) = (1, -1, -1),$$

hence ν^2 induces the isomorphism $SO(8) \cong Ss(8)$.

Exceptional Lie group F_4

2.1. Exceptional Jordan algebra \mathfrak{J}

Let $\mathfrak{J} = \mathfrak{J}(3, \mathfrak{C})$ denote all 3×3 Hermitian matrices with entries in the Cayley algebra \mathfrak{C} :

$$\mathfrak{J} = \{X \in M(3, \mathfrak{C}) \mid X^* = X\},$$

where $X^* = {}^t\overline{X}$. Any element $X \in \mathfrak{J}$ is of the form

$$X = X(\xi, x) = \begin{pmatrix} \xi_1 & x_3 & \overline{x}_2 \\ \overline{x}_3 & \xi_2 & x_1 \\ x_2 & \overline{x}_1 & \xi_3 \end{pmatrix}, \quad \xi_i \in \mathbf{R}, x_i \in \mathfrak{C}.$$

\mathfrak{J} is a 27 dimensional \mathbf{R} -vector space. In \mathfrak{J} , the multiplication $X \circ Y$, called the Jordan multiplication, is defined by

$$X \circ Y = \frac{1}{2}(XY + YX).$$

In \mathfrak{J} , we define the trace $\text{tr}(X)$, an inner product (X, Y) and a trilinear form $\text{tr}(X, Y, Z)$ respectively by

$$\begin{aligned} \text{tr}(X) &= \xi_1 + \xi_2 + \xi_3, & X &= X(\xi, x), \\ (X, Y) &= \text{tr}(X \circ Y), & \text{tr}(X, Y, Z) &= (X, Y \circ Z). \end{aligned}$$

Moreover, in \mathfrak{J} , we define a multiplication $X \times Y$, called the Freudenthal multiplication, by

$$X \times Y = \frac{1}{2}(2X \circ Y - \text{tr}(X)Y - \text{tr}(Y)X + (\text{tr}(X)\text{tr}(Y) - (X, Y))E),$$

(where E is the 3×3 unit matrix) and a trilinear form (X, Y, Z) and the determinant $\det X$ respectively by

$$(X, Y, Z) = (X, Y \times Z), \quad \det X = \frac{1}{3}(X, X, X).$$

For $X = X(\xi, x)$, $Y = Y(\eta, y)$ and $Z = Z(\zeta, z) \in \mathfrak{J}$, the explicit forms in the terms of their entries are as follows.

$$\begin{aligned} (X, Y) &= \sum_{i=1}^3 (\xi_i \eta_i + 2(x_i, y_i)), \\ \text{tr}(X, Y, Z) &= \sum_{i=1}^3 (\xi_i \eta_i \zeta_i + R(x_i y_{i+1} z_{i+2} + x_i z_{i+1} y_{i+2}) \\ &\quad + \xi_i((y_{i+1}, z_{i+1}) + (y_{i+2}, z_{i+2})) + \eta_i((z_{i+1}, x_{i+1}) + (z_{i+2}, x_{i+2})) \\ &\quad + \zeta_i((x_{i+1}, y_{i+1}) + (x_{i+2}, y_{i+2}))), \end{aligned}$$

$$\begin{aligned}
(X, Y, Z) &= \sum_{i=1}^3 \left(\frac{1}{2} (\xi_i \eta_{i+1} \zeta_{i+2} + \xi_i \eta_{i+2} \zeta_{i+1}) + R(x_i y_{i+1} z_{i+2} + x_i z_{i+1} y_{i+2}) \right. \\
&\quad \left. - (\xi_i(y_i, z_i) + \eta_i(z_i, x_i) + \zeta_i(x_i, y_i)) \right), \\
\det X &= \xi_1 \xi_2 \xi_3 + 2R(x_1 x_2 x_3) - \xi_1 x_1 \bar{x}_1 - \xi_2 x_2 \bar{x}_2 - \xi_3 x_3 \bar{x}_3.
\end{aligned}$$

Lemma 2.1.1. *The followings hold in \mathfrak{J} .*

- (1) (i) $X \circ Y = Y \circ X, \quad X \times Y = Y \times X.$
(ii) $E \circ X = X, \quad E \times X = \frac{1}{2}(\text{tr}(X)E - X), \quad E \times E = E.$
- (2) (i) *The inner product (X, Y) is symmetric and positive definite.*
(ii) $\text{tr}(X, Y, Z) = \text{tr}(Y, Z, X) = \text{tr}(Z, X, Y) = \text{tr}(X, Z, Y) = \text{tr}(Y, X, Z) = \text{tr}(Z, Y, X).$
The similar statement is also valid for (X, Y, Z) .
- (iii) $(X, E) = (X, E, E) = \text{tr}(X, E, E) = \text{tr}(X), \quad \text{tr}(X, Y, E) = (X, Y).$
(iv) $\text{tr}(X \times Y) = \frac{1}{2}(\text{tr}(X)\text{tr}(Y) - (X, Y)).$
- (3) (i) $(X \times X) \circ X = (\det X)E$ (Hamilton-Cayley).
(ii) $(X \times X) \times (X \times X) = (\det X)X.$

Proof. (1) is evident.

(2) is clear from the explicit forms of $(X, Y), \text{tr}(X, Y, Z), (X, Y, Z)$ etc.

(3) Using the following explicit form

$$X \times X = \begin{pmatrix} \xi_2 \xi_3 - x_1 \bar{x}_1 & \bar{x}_1 \bar{x}_2 - \xi_3 x_3 & x_3 x_1 - \xi_2 \bar{x}_2 \\ x_1 x_2 - \xi_3 \bar{x}_3 & \xi_3 \xi_1 - x_2 \bar{x}_2 & \bar{x}_2 \bar{x}_3 - \xi_1 x_1 \\ \bar{x}_3 \bar{x}_1 - \xi_2 x_2 & x_2 x_3 - \xi_1 \bar{x}_1 & \xi_1 \xi_2 - x_3 \bar{x}_3 \end{pmatrix}, \quad X = X(\xi, x),$$

each formula is obtained by direct calculations.

In \mathfrak{J} , we adopt the following notations:

$$\begin{aligned}
E_1 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\
F_1(x) &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & x \\ 0 & \bar{x} & 0 \end{pmatrix}, \quad F_2(x) = \begin{pmatrix} 0 & 0 & \bar{x} \\ 0 & 0 & 0 \\ x & 0 & 0 \end{pmatrix}, \quad F_3(x) = \begin{pmatrix} 0 & x & 0 \\ \bar{x} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\end{aligned}$$

The tables of the Jordan multiplication and the Freudenthal multiplication among elements above are given as follows.

$$\left\{ \begin{array}{l} E_i \circ E_i = E_i \\ E_i \circ F_i(x) = 0 \\ F_i(x) \circ F_i(y) = (x, y)(E_{i+1} + E_{i+2}), \end{array} \right. \quad \left\{ \begin{array}{l} E_i \circ E_j = 0, \quad i \neq j \\ E_i \circ F_j(x) = \frac{1}{2}F_j(x), \quad i \neq j \\ F_i(x) \circ F_{i+1}(y) = \frac{1}{2}F_{i+2}(\overline{xy}), \end{array} \right.$$

$$\left\{ \begin{array}{l} E_i \times E_i = 0 \\ E_i \times F_i(x) = -\frac{1}{2}F_i(x) \\ F_i(x) \times F_i(y) = -(x, y)E_i, \end{array} \right. \quad \left\{ \begin{array}{l} E_i \times E_{i+1} = \frac{1}{2}E_{i+2} \\ E_i \times F_j(x) = 0, \quad i \neq j \\ F_i(x) \times F_{i+1}(y) = \frac{1}{2}F_{i+2}(\overline{xy}), \end{array} \right.$$

where the indexes are considered as mod 3.

2.2. Compact exceptional Lie group F_4

Definition. The group F_4 is defined to be the automorphism group of the Jordan algebra \mathfrak{J} :

$$F_4 = \{\alpha \in \text{Iso}_{\mathbf{R}}(\mathfrak{J}) \mid \alpha(X \circ Y) = \alpha X \circ \alpha Y\}.$$

Lemma 2.2.1. (1) For $\alpha \in F_4$, we have $\alpha E = E$.

(2) For $\alpha \in F_4$, we have $\text{tr}(\alpha X) = \text{tr}(X)$, $X \in \mathfrak{J}$.

Proof. (1) Applying α on $E \circ X = X$, we have $\alpha E \circ \alpha X = \alpha X$. Let $X = \alpha^{-1}E$, then $\alpha E \circ E = E$, that is, $\alpha E = E$.

(2) We use the Hamilton-Cayley identity $X \circ (X \times X) = (\det X)E$ (Lemma 2.1.1.(3)), that is,

$$X \circ (X \circ X) - \text{tr}(X)X^2 + \frac{1}{2}(\text{tr}(X)^2 - \text{tr}(X^2))X = (\det X)E. \quad (\text{i})$$

We put αX in the place of X of (i) and then apply $\alpha^{-1} \in F_4$ on the obtained expression. Then

$$X \circ (X \circ X) - \text{tr}(\alpha X)X^2 + \frac{1}{2}(\text{tr}(\alpha X)^2 - \text{tr}((\alpha X)^2))X = (\det \alpha X)E. \quad (\text{ii})$$

By subtracting (i)–(ii), we get

$$\begin{aligned} & (\text{tr}(\alpha X) - \text{tr}(X))X^2 + \frac{1}{2}(\text{tr}(X)^2 - \text{tr}(\alpha X)^2 + \text{tr}((\alpha X)^2) - \text{tr}(X^2))X \\ & = (\det X - \det(\alpha X))E. \end{aligned}$$

Let $X = F_i(e_j)$, $i = 1, 2, 3, j = 0, 1, \dots, 7$, then

$$\begin{aligned} & \text{tr}(\alpha F_i(e_j))(E_{i+1} + E_{i+2}) + \frac{1}{2}(-\text{tr}(\alpha F_i(e_j))^2 + \text{tr}((\alpha F_i(e_j))^2) - 2)F_i(e_j) \\ & = -\det(\alpha F_i(e_j))E. \end{aligned}$$

Comparing the entries of both sides of the equation above, we have

$$\text{tr}(\alpha F_i(e_j)) = 0 \quad (= \text{tr}(F_i(e_j)))$$

and $\text{tr}((\alpha F_i(e_j))^2) = 2$, hence

$$\text{tr}(\alpha E_i) = \text{tr}(\alpha(E - F_i(1)^2)) = \text{tr}(E) - \text{tr}((\alpha F_i(1))^2) = 3 - 2 = 1 = \text{tr}(E_i),$$

for $i = 1, 2, 3$. Consequently, we have $\text{tr}(\alpha X) = \text{tr}(X)$ for every $X = E_i, F_i(e_j)$ of the \mathbf{R} -basis of \mathfrak{J} . Thus the lemma is proved.

For $\alpha \in \text{Hom}_{\mathbf{R}}(\mathfrak{J})$, we denote by ${}^t\alpha$ the transpose of α with respect to the inner product (X, Y) : $({}^t\alpha X, Y) = (X, \alpha Y)$.

Lemma 2.2.2. *For $\alpha \in \text{Iso}_{\mathbf{R}}(\mathfrak{J})$, the following four conditions are equivalent.*

- (1) $\det(\alpha X) = \det X$, for all $X \in \mathfrak{J}$.
- (2) $(\alpha X, \alpha Y, \alpha Z) = (X, Y, Z)$, for all $X, Y, Z \in \mathfrak{J}$.
- (3) $\alpha X \times \alpha Y = {}^t\alpha^{-1}(X \times Y)$, for all $X, Y \in \mathfrak{J}$.
- (4) $\alpha X \times \alpha X = {}^t\alpha^{-1}(X \times X)$, for all $X \in \mathfrak{J}$.

Proof. (1) \Rightarrow (2) $\det(\alpha X) = \det X$ implies that $(\alpha X, \alpha X, \alpha X) = (X, X, X)$. Putting $\lambda X + \mu Y + \nu Z$ in place of X and comparing the coefficient of $\lambda\mu\nu$, we obtain (2).

(2) \Rightarrow (1) is evident.

(4) \Rightarrow (3) Putting $\lambda X + \mu Y$ in place of X and comparing the coefficient of $\lambda\mu$, we obtain (3).

(3) \Rightarrow (4) is evident.

(2) \Leftrightarrow (3) is easily obtained.

Lemma 2.2.3. *If $\alpha \in \text{Iso}_{\mathbf{R}}(\mathfrak{J})$ satisfies $\det(\alpha X) = \det X$ for all $X \in \mathfrak{J}$, then we have*

$$\det({}^t\alpha^{-1}X) = \det({}^t\alpha X) = \det X, \quad \text{for all } X \in \mathfrak{J}.$$

Proof. We have

$$\begin{aligned} {}^t\alpha^{-1}(Y \times Y) \times {}^t\alpha^{-1}(Y \times Y) &= (\alpha Y \times \alpha Y) \times (\alpha Y \times \alpha Y) \quad (\text{Lemma 2.2.2.(4)}) \\ &= (\det \alpha Y) \alpha Y \quad (\text{Lemma 2.1.1.(3)}) = (\det Y) \alpha Y = \alpha((\det Y)Y) \\ &= \alpha((Y \times Y) \times (Y \times Y)), \quad Y \in \mathfrak{J}. \end{aligned}$$

Let $Y = X \times X$, $X \in \mathfrak{J}$ in the above, then

$${}^t\alpha^{-1}((\det X)X) \times {}^t\alpha^{-1}((\det X)X) = \alpha((\det X)X \times (\det X)X).$$

We now consider the following two cases.

- (1) Case $\det X \neq 0$. In this case, we have ${}^t\alpha^{-1}X \times {}^t\alpha^{-1}X = \alpha(X \times X)$. Hence,

$$\begin{aligned} 3\det({}^t\alpha^{-1}X) &= ({}^t\alpha^{-1}X, {}^t\alpha^{-1}X \times {}^t\alpha^{-1}X) \\ &= ({}^t\alpha^{-1}X, \alpha(X \times X)) = (X, X \times X) = 3\det X, \end{aligned}$$

hence we have $\det({}^t\alpha^{-1}X) = \det X$. Next, if we use α^{-1} instead of α , we can see also that $\det({}^t\alpha X) = \det X$.

(2) Case $\det X = 0$. If $\det({}^t\alpha^{-1}X) \neq 0$, we can use the result of (1). If we put ${}^t\alpha X$ instead of X in $\det({}^t\alpha^{-1}X) = \det X$ of (1), then $\det({}^t\alpha^{-1}({}^t\alpha X)) = \det({}^t\alpha X)$. Hence $0 = \det X = \det({}^t\alpha X) \neq 0$ which is a contradiction. Thus we have $\det({}^t\alpha^{-1}X) = \det({}^t\alpha X) = 0$, so $\det({}^t\alpha^{-1}X) = \det({}^t\alpha X) = \det X$ is also valid.

Lemma 2.2.4. *For $\alpha \in \text{Iso}_{\mathbf{R}}(\mathfrak{J})$, the following five conditions are equivalent.*

- (1) $\alpha(X \circ Y) = \alpha X \circ \alpha Y$.
- (2) $\text{tr}(\alpha X, \alpha Y, \alpha Z) = \text{tr}(X, Y, Z)$, $(\alpha X, \alpha Y) = (X, Y)$.
- (3) $\det(\alpha X) = \det X$, $(\alpha X, \alpha Y) = (X, Y)$.
- (4) $\det(\alpha X) = \det X$, $\alpha E = E$.
- (5) $\alpha(X \times Y) = \alpha X \times \alpha Y$.

Proof. (1) \Rightarrow (2) $(\alpha X, \alpha Y) = \text{tr}(\alpha X \circ \alpha Y) = \text{tr}(\alpha(X \circ Y)) = \text{tr}(X \circ Y)$ (Lemma 2.2.1) $= (X, Y)$. Also $\text{tr}(\alpha X, \alpha Y, \alpha Z) = (\alpha X, \alpha Y \circ \alpha Z) = (\alpha X, \alpha(Y \circ Z)) = (X, Y \circ Z) = \text{tr}(X, Y, Z)$.

(2) \Rightarrow (1) $(\alpha X \circ \alpha Y, \alpha Z) = \text{tr}(\alpha X, \alpha Y, \alpha Z) = \text{tr}(X, Y, Z) = (X \circ Y, Z) = (\alpha(X \circ Y), \alpha Z)$ holds for all αZ , so we have $\alpha X \circ \alpha Y = \alpha(X \circ Y)$.

(2) \Rightarrow (3) Since we have already shown (2) \Rightarrow (1), we can use $\text{tr}(\alpha X) = \text{tr}(X)$ (Lemma 2.2.1.(2)). Now,

$$\begin{aligned} 3\det(\alpha X) &= \text{tr}(\alpha X, \alpha X, \alpha X) - \frac{3}{2}\text{tr}(\alpha X)(\alpha X, \alpha X) + \frac{1}{2}\text{tr}(\alpha X)^3 \\ &= \text{tr}(X, X, X) - \frac{3}{2}\text{tr}(X)(X, X) + \frac{1}{2}\text{tr}(X)^3 = 3\det X. \end{aligned}$$

(3) \Rightarrow (5) $(\alpha(X \times Y), \alpha Z) = (X \times Y, Z) = (X, Y, Z) = (\alpha X, \alpha Y, \alpha Z)$ (Lemma 2.2.2) $= (\alpha X \times \alpha Y, \alpha Z)$ holds for all $\alpha Z \in \mathfrak{J}$, so we have $\alpha X \times \alpha Y = \alpha(X \times Y)$.

$$\begin{aligned} (5) \Rightarrow (4) \quad (\det(\alpha X))\alpha X &= (\alpha X \times \alpha X) \times (\alpha X \times \alpha X) \quad (\text{Lemma 2.1.1.(3)}) \\ &= \alpha((X \times X) \times (X \times X)) = (\det X)\alpha X \quad (\text{Lemma 2.1.1.(3)}) \end{aligned}$$

and so we have $\det(\alpha X) = \det X$. In the relation

$$\alpha X \circ \alpha E = \alpha(X \times E) = \frac{1}{2}\alpha(\text{tr}(X)E - X),$$

if we denote $\alpha E = P$, then

$$\alpha X \times P = \frac{1}{2}\text{tr}(X)P - \frac{1}{2}\alpha X, \quad X \in \mathfrak{J}.$$

Let $X = \alpha^{-1}E_1$ and $P = \rho_1 E_1 + \rho_2 E_2 + \rho_3 E_3 + F_1(p_1) + F_2(p_2) + F_3(p_3)$, then

$$\begin{aligned} &\frac{1}{2}(\rho_2 E_3 + \rho_3 E_2 - F_1(p_1)) \\ &= \frac{1}{2}(\lambda(\rho_1 E_1 + \rho_2 E_2 + \rho_3 E_3 + F_1(p_1) + F_2(p_2) + F_3(p_3)) - E_1), \end{aligned}$$

where $\lambda = \text{tr}(\alpha^{-1}E_1)$. By comparing entries of both sides, we have

$$0 = \lambda\rho_1 - 1, \quad \rho_3 = \lambda\rho_2, \quad \rho_2 = \lambda\rho_3, \quad -p_1 = \lambda p_1, \quad 0 = \lambda p_2, \quad 0 = \lambda p_3.$$

Consequently we have $p_2 = p_3 = 0$. Similarly, by letting $X = \alpha^{-1}E_2$, we also have $p_1 = 0$. Again put $X = \alpha^{-1}F_1(1)$, then

$$-\frac{1}{2}\rho_1 F_1(1) = \frac{1}{2}(\mu(\rho_1 E_1 + \rho_2 E_2 + \rho_3 E_3) - F_1(1)),$$

where $\mu = \text{tr}(\alpha^{-1}F_1(1))$. By Comparing entries of F_1 -parts, we see that $\rho_1 = 1$. Similarly $\rho_2 = \rho_3 = 1$. Therefore we have $\alpha E = E$.

(4) \Rightarrow (2) $\text{tr}(\alpha X) = (\alpha X, E, E) = (\alpha X, \alpha E, \alpha E) = (X, E, E)$ (Lemma 2.2.2.(2))
 $= \text{tr}(X)$. Hence $\frac{1}{2}(\text{tr}(X)\text{tr}(Y) - (X, Y)) = (X, Y, E)$ (Lemma 2.1.1) $= (\alpha X, \alpha Y, \alpha E)$
 $= (\alpha X, \alpha Y, E) = \frac{1}{2}(\text{tr}(\alpha X) \text{tr}(\alpha Y) - (\alpha X, \alpha Y)) = \frac{1}{2}(\text{tr}(X)\text{tr}(Y) - (\alpha X, \alpha Y))$. Therefore we obtain

$$(\alpha X, \alpha Y) = (X, Y).$$

Next, using the relation $(X, Y, Z) = \text{tr}(X, Y, Z) - \frac{1}{2}\text{tr}(X)(Y, Z) - \frac{1}{2}\text{tr}(Y)(Z, X) - \frac{1}{2}\text{tr}(Z)(X, Y) + \frac{1}{2}\text{tr}(X)\text{tr}(Y)\text{tr}(Z)$ and $(\alpha X, \alpha Y, \alpha Z) = (X, Y, Z)$, we obtain

$$\text{tr}(\alpha X, \alpha Y, \alpha Z) = \text{tr}(X, Y, Z).$$

Theorem 2.2.5. F_4 is a compact Lie group.

Proof. F_4 is a compact Lie group as a closed subgroup of the orthogonal group

$$O(27) = O(\mathfrak{J}) = \{\alpha \in \text{Iso}_{\mathbf{R}}(\mathfrak{J}) \mid (\alpha X, \alpha Y) = (X, Y)\} \quad (\text{Lemma 2.2.4.(2)}).$$

The group F_4 contains G_2 as a subgroup in the following way. For $\alpha \in G_2$, we consider a mapping $\tilde{\alpha} : \mathfrak{J} \rightarrow \mathfrak{J}$,

$$\tilde{\alpha} \begin{pmatrix} \xi_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & \xi_2 & x_1 \\ x_2 & \bar{x}_1 & \xi_3 \end{pmatrix} = \begin{pmatrix} \xi_1 & \alpha x_3 & \overline{\alpha x_2} \\ \overline{\alpha x_3} & \xi_2 & \alpha x_1 \\ \alpha x_2 & \overline{\alpha x_1} & \xi_3 \end{pmatrix}.$$

Then $\tilde{\alpha} \in F_4$. So we identify $\alpha \in G_2$ with $\tilde{\alpha} \in F_4$: $G_2 \subset F_4$.

2.3. Lie algebra \mathfrak{f}_4 of F_4

In order to investigate the Lie algebra \mathfrak{f}_4 of the group F_4 , it will be helpful to study the Lie algebra $\mathfrak{e}_{6(-26)}$ of the group

$$\begin{aligned} E_{6(-26)} &= \{\alpha \in \text{Iso}_{\mathbf{R}}(\mathfrak{J}) \mid \det(\alpha X) = \det X\} \\ &= \{\alpha \in \text{Iso}_{\mathbf{R}}(\mathfrak{J}) \mid {}^t\alpha^{-1}(X \times Y) = \alpha X \times \alpha Y\}. \end{aligned}$$

Lemma 2.3.1. *The Lie algebra $\mathfrak{e}_{6(-26)}$ of the group $E_{6(-26)}$ is given by*

$$\begin{aligned}\mathfrak{e}_{6(-26)} &= \{\phi \in \text{Hom}_{\mathbf{R}}(\mathfrak{J}) \mid (\phi X, X, X) = 0\} \\ &= \{\phi \in \text{Hom}_{\mathbf{R}}(\mathfrak{J}) \mid (\phi X, Y, Z) + (X, \phi Y, Z) + (X, Y, \phi Z) = 0\} \\ &= \{\phi \in \text{Hom}_{\mathbf{R}}(\mathfrak{J}) \mid -{}^t\phi(X \times Y) = \phi X \times Y + X \times \phi Y\}.\end{aligned}$$

Proof. It is easy to verify that these conditions in $\mathfrak{e}_{6(-26)}$ are equivalent (see Lemma 2.2.2). Now, if $\phi \in \text{Hom}_{\mathbf{R}}(\mathfrak{J})$ satisfies $((\exp t\phi)X, (\exp t\phi)X, (\exp t\phi)X) = (X, X, X)$ for all $t \in \mathbf{R}$, then we have $(\phi X, X, X) = 0$ by putting $t = 0$ after differentiating with respect to t . Conversely, if $\phi \in \text{Hom}_{\mathbf{R}}(\mathfrak{J})$ satisfies $-{}^t\phi(X \times Y) = \phi X \times Y + X \times \phi Y$, then it is easy to verify that $\alpha = \exp t\phi$ satisfies ${}^t\alpha^{-1}(X \times Y) = \alpha X \times \alpha Y$.

Theorem 2.3.2. *The Lie algebra \mathfrak{f}_4 of the group F_4 is given by*

$$\begin{aligned}\mathfrak{f}_4 &= \{\delta \in \text{Hom}_{\mathbf{R}}(\mathfrak{J}) \mid \delta(X \circ Y) = \delta X \circ Y + X \circ \delta Y\} \\ &= \left\{ \delta \in \text{Hom}_{\mathbf{R}}(\mathfrak{J}) \mid \begin{array}{l} \text{tr}(\delta X, Y, Z) + \text{tr}(X, \delta Y, Z) + \text{tr}(X, Y, \delta Z) = 0 \\ (\delta X, Y) + (X, \delta Y) = 0 \end{array} \right\} \\ &= \{\delta \in \text{Hom}_{\mathbf{R}}(\mathfrak{J}) \mid (\delta X, X, X) = 0, (\delta X, Y) + (X, \delta Y) = 0\} \\ &= \{\delta \in \text{Hom}_{\mathbf{R}}(\mathfrak{J}) \mid (\delta X, X, X) = 0, \delta E = 0\} \\ &= \{\delta \in \text{Hom}_{\mathbf{R}}(\mathfrak{J}) \mid \delta(X \times Y) = \delta X \times Y + X \times \delta Y\}.\end{aligned}$$

Proof. The theorem follows easily from Lemma 2.2.4.

We define an \mathbf{R} -vector space \mathfrak{M}^- by

$$\mathfrak{M}^- = \{A \in M(3, \mathfrak{C}) \mid A^* = -A\}.$$

For $X, Y \in M(3, \mathfrak{C})$, we define $[X, Y] \in M(3, \mathfrak{C})$ by

$$[X, Y] = XY - YX.$$

Lemma 2.3.3. $[\mathfrak{M}^-, \mathfrak{J}] \subset \mathfrak{J}, \quad [\mathfrak{J}, \mathfrak{J}] \subset \mathfrak{M}^-.$

Since $[\mathfrak{M}^-, \mathfrak{J}] \subset \mathfrak{J}$, any element $A \in \mathfrak{M}^-$ induces an \mathbf{R} -linear mapping $\tilde{A} : \mathfrak{J} \rightarrow \mathfrak{J}$ defined by

$$\tilde{A}X = \frac{1}{2}[A, X], \quad X \in \mathfrak{J}.$$

Lemma 2.3.4. *For $X \in \mathfrak{J}$, there exists $a \in \mathfrak{C}_0$ such that*

$$[X, XX] = aE.$$

Proof. Let $X = (x_{ij})$, $x_{ij} \in \mathfrak{C}$, $\bar{x}_{ij} = x_{ji}$. The (i, j) -entry a_{ij} of $[X, XX] = X(XX) - (XX)X$ is given by

$$a_{ij} = \sum_{k,l} (x_{ik}(x_{kl}x_{lj}) - (x_{ik}x_{kl})x_{lj}) = - \sum_{k,l} \{x_{ik}, x_{kl}, x_{lj}\}, \quad i, j = 1, 2, 3.$$

Since x_{ii} is real, if the bracket $\{ \cdot, \cdot, \cdot \}$ contains x_{ii} , then $\{ \cdot, \cdot, \cdot \}$ is 0. If $i \neq j$, then a_{ij} is a sum of $a(xa) - (ax)a$, $a(\bar{a}x) - (a\bar{a})x$ and so on, so $a_{ij} = 0$. If $i = j$, then

$$\begin{aligned} a_{11} &= -\{x_{12}, x_{23}, x_{31}\} - \{x_{13}, x_{32}, x_{21}\}, \\ a_{22} &= -\{x_{21}, x_{13}, x_{32}\} - \{x_{23}, x_{31}, x_{12}\}, \\ a_{33} &= -\{x_{31}, x_{12}, x_{23}\} - \{x_{32}, x_{21}, x_{13}\}, \end{aligned}$$

however they are equal, that is, $a_{11} = a_{22} = a_{33} (= a)$. Hence we have $[X, XX] = aE$. Since $X, XX \in \mathfrak{J}$, we have $[X, XX] \in \mathfrak{M}^-$ (Lemma 2.3.3). Therefore, $(aE)^* = -aE$ and so $\bar{a}E = -aE$, which imply that $\bar{a} = -a$.

To prove the following Proposition 2.3.6, in $M(3, \mathfrak{C})$, we define a real valued symmetric inner product (X, Y) by

$$(X, Y) = \frac{1}{2} \text{tr}(XY + Y^*X^*).$$

Lemma 2.3.5. *The inner product (X, Y) of $M(3, \mathfrak{C})$ satisfies*

$$(XY, Z) = (YZ, X) = (ZX, Y) = (X, YZ) = (Y, ZX) = (Z, XY).$$

Proof. Let $X = (x_{ij})$, $Y = (y_{ij})$, $Z = (z_{ij})$. Then we have

$$\begin{aligned} (XY, Z) &= R(XY, Z) = \frac{1}{2} R(\text{tr}((XY)Z + Z^*(Y^*X^*))) \\ &= \frac{1}{2} R\left(\sum_{i,j,k} ((x_{ij}y_{jk})z_{ki} + \bar{z}_{ji}(\bar{y}_{kj}\bar{x}_{ik}))\right) \\ &= \frac{1}{2} R\left(\sum_{i,j,k} ((y_{jk}z_{ki})x_{ij} + \bar{x}_{ik}(\bar{z}_{ji}\bar{y}_{kj}))\right) \\ &= \frac{1}{2} R(\text{tr}((YZ)X + X^*(Z^*Y^*))) = (YZ, X). \end{aligned}$$

Proposition 2.3.6. *For $A \in \mathfrak{M}^-$, $\text{tr}(A) = 0$, we have $\tilde{A} \in \mathfrak{f}_4$.*

Proof. From the equivalent conditions in Theorem 2.3.2, it is sufficient to show the following two formulas:

$$\begin{cases} ([A, X], Y) + (X, [A, Y]) = 0, & X, Y \in \mathfrak{J}, \\ \text{tr}([A, X], Y, Z) + \text{tr}(X, [A, Y], Z) + \text{tr}(X, Y, [A, Z]) = 0, & X, Y, Z \in \mathfrak{J}. \end{cases}$$

Now, the left side of the first formula $= (AX, Y) - (XA, Y) + (X, AY) - (X, YA) = 0$ (Lemma 2.3.5). Next, we show that

$$([A, X], XX) = 0, \quad X \in \mathfrak{J}.$$

Certainly, if $[X, XX] = aE$, $a \in \mathfrak{C}_0$ (Lemma 2.3.4), then

$$\begin{aligned}
([A, X], XX) &= (AX, XX) - (XA, XX) \\
&= (A, X(XX)) - (A, (XX)X) \quad (\text{Lemma 2.3.5}) \\
&= (A, [X, XX]) = (A, aE) = \frac{1}{2}\text{tr}(Aa + \bar{a}A^*) \\
&= \frac{1}{2}\text{tr}(Aa - aA) = \frac{1}{2}(\text{tr}(A)a - a\text{tr}(A)) = 0.
\end{aligned}$$

Now, putting $\lambda X + \mu X + \nu Z$ in the place of X , and comparing the coefficient of $\lambda\mu\nu$, we obtain

$$([A, X], YZ + ZY) + ([A, Y], XZ + ZX) + ([A, Z], XY + YX) = 0,$$

which is the required second formula.

In \mathfrak{M}^- , we adopt the following notation:

$$A_1(a) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & a \\ 0 & -\bar{a} & 0 \end{pmatrix}, \quad A_2(a) = \begin{pmatrix} 0 & 0 & -\bar{a} \\ 0 & 0 & 0 \\ a & 0 & 0 \end{pmatrix}, \quad A_3(a) = \begin{pmatrix} 0 & a & 0 \\ -\bar{a} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then $\tilde{A}_i(a) \in \mathfrak{f}_4$ (Proposition 2.3.6) and the operation of $\tilde{A}_i(a)$ on \mathfrak{J} is given by

$$\begin{cases} \tilde{A}_i(a)E_i = 0 \\ \tilde{A}_i(a)E_{i+1} = -\frac{1}{2}F_i(a) \\ \tilde{A}_i(a)E_{i+2} = \frac{1}{2}F_i(a), \end{cases} \quad \begin{cases} \tilde{A}_i(a)F_i(x) = (a, x)(E_{i+1} - E_{i+2}) \\ \tilde{A}_i(a)F_{i+1}(x) = \frac{1}{2}F_{i+2}(\bar{a}x) \\ \tilde{A}_i(a)F_{i+2}(x) = -\frac{1}{2}F_{i+1}(\bar{x}a). \end{cases}$$

Proposition 2.3.7. *The Lie subalgebra \mathfrak{d}_4 of \mathfrak{f}_4 :*

$$\mathfrak{d}_4 = \{\delta \in \mathfrak{f}_4 \mid \delta E_i = 0, i = 1, 2, 3\}$$

is isomorphic to the Lie algebra $\mathfrak{D}_4 = \mathfrak{so}(8)$:

$$\mathfrak{D}_4 = \{D \in \text{Hom}_{\mathbf{R}}(\mathfrak{C}) \mid (Dx, y) + (x, Dy) = 0\}$$

under the correspondence

$$\mathfrak{D}_4 \ni D_1 \longrightarrow \delta \in \mathfrak{d}_4$$

given by

$$\delta \begin{pmatrix} \xi_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & \xi_2 & x_1 \\ x_2 & \bar{x}_1 & \xi_3 \end{pmatrix} = \begin{pmatrix} 0 & D_3 x_3 & \overline{D_2 x_2} \\ \overline{D_3 x_3} & 0 & D_1 x_1 \\ D_2 x_2 & \overline{D_1 x_1} & 0 \end{pmatrix},$$

where D_2, D_3 are elements of \mathfrak{D}_4 which are determined by D_1 from the principle of triality:

$$(D_1 x)y + x(D_2 y) = \overline{D_3(\bar{x}y)}, \quad x, y \in \mathfrak{C}.$$

(From now on, we identify these Lie algebras \mathfrak{d}_4 and \mathfrak{D}_4).

Proof. We define a mapping $\varphi_* : \mathfrak{D}_4 \rightarrow \mathfrak{d}_4$ by $\varphi_*(D_1) = \delta$. We first prove that $\delta \in \mathfrak{d}_4 \subset \mathfrak{f}_4$. Indeed,

$$\begin{aligned}
(\delta X, X, X) &= (\delta X, X \times X) \\
&= \left(\begin{pmatrix} 0 & D_3 x_3 & \overline{D_2 x_2} \\ \overline{D_3 x_3} & 0 & D_1 x_1 \\ D_2 x_2 & \overline{D_1 x_1} & 0 \end{pmatrix}, \begin{pmatrix} \xi_2 \xi_3 - x_1 \bar{x}_1 & \overline{x_1 x_2} - \xi_3 x_3 & x_3 x_1 - \xi_2 \bar{x}_2 \\ x_1 x_2 - \xi_3 \bar{x}_3 & \xi_3 \xi_1 - x_2 \bar{x}_2 & \overline{x_2 x_3} - \xi_1 x_1 \\ \overline{x_3 x_1} - \xi_2 x_2 & x_2 x_3 - \xi_1 \bar{x}_1 & \xi_1 \xi_2 - x_3 \bar{x}_3 \end{pmatrix} \right) \\
&= 2(D_1 x_1, \overline{x_2 x_3} - \xi_1 x_1) + 2(D_2 x_2, \overline{x_3 x_1} - \xi_2 x_2) + 2(D_3 x_3, \overline{x_1 x_2} - \xi_3 x_3) \\
&= 2((D_1 x_1, \overline{x_2 x_3}) + (D_2 x_2, \overline{x_3 x_1}) + (D_3 x_3, \overline{x_1 x_2})) \\
&= 2((D_1 x_1, \overline{x_2 x_3}) + (\overline{D_2 x_2}, x_3 x_1) + (\overline{D_3 x_3}, x_1 x_2))
\end{aligned}$$

which is equal to 0, if we use the relation

$$\begin{aligned}
(D_1 x_1, \overline{x_2 x_3}) &= -(x_1, D_1(\overline{x_2 x_3})) = -(x_1, \overline{(D_2 x_2)x_3 + x_2(D_3 x_3)}) \\
&= -(x_1, \overline{x_3(D_2 x_2)} + \overline{(D_3 x_3)x_2}) = -(x_3 x_1, \overline{D_2 x_2}) - (x_1 x_2, \overline{D_3 x_3}).
\end{aligned}$$

Hence $\delta \in \mathfrak{f}_4$. $\delta E_i = 0, i = 1, 2, 3$ are evident, so that $\delta \in \mathfrak{d}_4$. Clearly φ_* is injective. We shall show that φ_* is onto. Let $\delta \in \mathfrak{d}_4$. We put

$$\mathfrak{J}_i = \{F_i(x) \mid x \in \mathfrak{C}\} = \{X \in \mathfrak{J} \mid 2E_{i+1} \circ X = 2E_{i+2} \circ X = X\}.$$

Since $\delta X \in \mathfrak{J}_i$ for $X \in \mathfrak{J}_i$, δ induces \mathbf{R} -linear mappings $\delta : \mathfrak{J}_i \rightarrow \mathfrak{J}_i$ and $D_i : \mathfrak{C} \rightarrow \mathfrak{C}$ satisfying

$$\delta F_i(x) = F_i(D_i x), \quad x \in \mathfrak{C},$$

for $i = 1, 2, 3$. Applying δ on $F_i(x) \circ F_i(y) = (x, y)(E_{i+1} + E_{i+2})$, we have $F_i(D_i x) \circ F_i(y) + F_i(x) \circ F_i(D_i y) = 0$, and hence we have

$$(D_i x, y) + (x, D_i y) = 0, \quad x, y \in \mathfrak{C}.$$

Hence $D_i \in \mathfrak{D}_4, i = 1, 2, 3$. Moreover, by applying δ on $F_1(x) \circ F_2(y) = \frac{1}{2}F_3(\overline{xy})$, we see that

$$(D_1 x)y + x(D_2 y) = \overline{D_3(\overline{xy})}, \quad x, y \in \mathfrak{C}.$$

This shows that φ_* is onto. Thus Proposition 2.3.7 is proved.

Theorem 2.3.8. Any element $\delta \in \mathfrak{f}_4$ is uniquely expressed by

$$\delta = D + \tilde{A}, \quad D \in \mathfrak{d}_4, A \in \mathfrak{M}^-, \text{diag} A = 0,$$

where $\text{diag} A = 0$ means that all diagonal elements a_{ii} of A are 0. In particular, the dimension of \mathfrak{f}_4 is

$$\dim \mathfrak{f}_4 = 28 + 24 = 52.$$

Proof. Applying $\delta \in \mathfrak{f}_4$ on $E_i \circ E_i = E_i$ and $E_i \circ E_j = 0, i \neq j$, we have

$$2\delta E_i \circ E_i = \delta E_i, \quad \delta E_i \circ E_j + E_i \circ \delta E_j = 0.$$

From these relations, we see that each δE_i is of the form

$$\delta E_1 = \begin{pmatrix} 0 & -a_3 & \bar{a}_2 \\ -\bar{a}_3 & 0 & 0 \\ a_2 & 0 & 0 \end{pmatrix}, \delta E_2 = \begin{pmatrix} 0 & a_3 & 0 \\ \bar{a}_3 & 0 & -a_1 \\ 0 & -\bar{a}_1 & 0 \end{pmatrix}, \delta E_3 = \begin{pmatrix} 0 & 0 & -\bar{a}_2 \\ 0 & 0 & a_1 \\ -a_2 & \bar{a}_1 & 0 \end{pmatrix},$$

$a_i \in \mathfrak{C}$. If we construct a matrix $A = 2 \begin{pmatrix} 0 & a_3 & -\bar{a}_2 \\ -\bar{a}_3 & 0 & a_1 \\ a_2 & -\bar{a}_1 & 0 \end{pmatrix}$ using these elements a_i ,

then $A \in \mathfrak{M}^-$ with $\text{diag} A = 0$, and we have

$$\delta E_i = \tilde{A} E_i, \quad i = 1, 2, 3.$$

If we put $D = \delta - \tilde{A}$, then $DE_i = 0$, $i = 1, 2, 3$, hence $D \in \mathfrak{d}_4$. Thus, δ can be expressed by $\delta = D + \tilde{A}$, where $D \in \mathfrak{d}_4$, $A \in \mathfrak{M}^-$, $\text{diag} A = 0$. To show the uniqueness of the expression, it is sufficient to prove that

$$D + \tilde{A} = 0, \quad D \in \mathfrak{d}_4, A \in \mathfrak{M}^-, \text{diag} A = 0, \quad \text{then } D = 0, A = 0.$$

Certainly, if we apply it on E_i , then $\tilde{A} E_i = 0$, $i = 1, 2, 3$, hence $A = 0$ and so $D = 0$. Finally, we have $\dim \mathfrak{f}_4 = 28 + 24 = 52$ from the expression above. Thus the theorem is proved.

2.4. Simplicity of \mathfrak{f}_4^C

Let $\mathfrak{J}^C = \{X_1 + iX_2 \mid X_1, X_2 \in \mathfrak{J}\}$ be the complexification of the Jordan algebra \mathfrak{J} . In the same manner as in \mathfrak{J} , in \mathfrak{J}^C we can also define the multiplications $X \circ Y$, $X \times Y$, the inner product (X, Y) , the trilinear forms $\text{tr}(X, Y, Z)$, (X, Y, Z) and the determinant $\det X$. They have the same properties as those of \mathfrak{J} . \mathfrak{J}^C is called the complex exceptional Jordan algebra. \mathfrak{J}^C has two complex conjugations, namely,

$$\overline{X_1 + iX_2} = \overline{X_1} + i\overline{X_2}, \quad \tau(X_1 + iX_2) = X_1 - iX_2, \quad X_i \in \mathfrak{J}.$$

The complex conjugation τ of \mathfrak{J}^C satisfies

$$\tau(X \circ Y) = \tau X \circ \tau Y, \quad \tau(X \times Y) = \tau X \times \tau Y, \quad X, Y \in \mathfrak{J}^C.$$

We define Lie algebras \mathfrak{e}_6^C and \mathfrak{f}_4^C respectively by

$$\begin{aligned} \mathfrak{e}_6^C &= \{\phi \in \text{Hom}_C(\mathfrak{J}^C) \mid (\phi X, X, X) = 0\} \\ &= \{\phi \in \text{Hom}_C(\mathfrak{J}^C) \mid -{}^t\phi(X \times Y) = \phi X \times Y + X \times \phi Y\}, \\ \mathfrak{f}_4^C &= \{\delta \in \text{Hom}_C(\mathfrak{J}^C) \mid \delta(X \circ Y) = \delta X \circ Y + X \circ \delta Y\} \\ &= \{\delta \in \text{Hom}_C(\mathfrak{J}^C) \mid \delta(X \times Y) = \delta X \times Y + X \times \delta Y\}. \end{aligned}$$

These are the complexification of the Lie algebras $\mathfrak{e}_{6(-26)}$ and \mathfrak{f}_4 respectively. Then the properties of $\mathfrak{e}_{6(-26)}$ and \mathfrak{f}_4 stated in Section 2.3 also hold for \mathfrak{e}_6^C and \mathfrak{f}_4^C . Hereafter we will describe \mathfrak{f}_4^C , but the statements are also valid for \mathfrak{f}_4 using \mathfrak{J} instead of \mathfrak{J}^C .

For $A \in \mathfrak{J}^C$, we define a C -linear mapping $\tilde{A} : \mathfrak{J}^C \rightarrow \mathfrak{J}^C$ by

$$\tilde{A}X = A \circ X, \quad X \in \mathfrak{J}^C.$$

Proposition 2.4.1. (1) For $A \in \mathfrak{J}^C$, $\text{tr}(A) = 0$, we have $\tilde{A} \in \mathfrak{e}_6^C$.

(2) For $A, B \in \mathfrak{J}^C$, we have $[\tilde{A}, \tilde{B}] \in \mathfrak{f}_4^C$.

Proof. (1) $(\tilde{A}X, X, X) = (A \circ X, X \times X) = (A, X \circ (X \times X))$
 $= (A, (\det X)E) = (\det X)(A, E) = (\det X)\text{tr}(A) = 0, \quad X \in \mathfrak{J}^C.$

Hence $\tilde{A} \in \mathfrak{e}_6^C$.

$$(2) \quad [\tilde{A}, \tilde{B}] = \left[\left(A - \frac{1}{3}\text{tr}(A)E \right)^\sim, \left(B - \frac{1}{3}\text{tr}(B)E \right)^\sim \right] \in \mathfrak{e}_6^C,$$

$$[\tilde{A}, \tilde{B}]E = \tilde{A}(\tilde{B}E) - \tilde{B}(\tilde{A}E) = A \circ B - B \circ A = 0.$$

Hence $[\tilde{A}, \tilde{B}] \in \mathfrak{f}_4^C$ (Theorem 2.3.2).

Lemma 2.4.2. In \mathfrak{f}_4^C , we have

$$\begin{aligned} [\tilde{E}_i, \tilde{E}_j] &= 0, & [\tilde{E}_i, \tilde{F}_i(a)] &= 0, \\ [\tilde{E}_i, \tilde{F}_{i+1}(a)] &= -\frac{1}{2}\tilde{A}_{i+1}(a), & [\tilde{E}_i, \tilde{F}_{i+2}(a)] &= \frac{1}{2}\tilde{A}_{i+2}(a), \\ [\tilde{F}_i(a), \tilde{F}_i(b)] &\in \mathfrak{d}_4^C, & [\tilde{F}_i(a), \tilde{F}_{i+1}(b)] &= -\frac{1}{2}\tilde{A}_{i+2}(\overline{ab}), \\ [D, \tilde{A}_i(a)] &= \tilde{A}_i(D_i a), & \text{where } D &= (D_1, D_2, D_3) \in \mathfrak{d}_4^C, \\ [\tilde{A}_i(a), \tilde{A}_i(b)] &\in \mathfrak{d}_4^C, & [\tilde{A}_i(a), \tilde{A}_{i+1}(b)] &= -\frac{1}{2}\tilde{A}_{i+2}(\overline{ab}), \\ [\tilde{F}_1(e_i), \tilde{F}_1(e_j)] &= G_{ij}, & [\tilde{A}_i(a), \tilde{A}_{i+2}(b)] &= \frac{1}{2}\tilde{A}_{i+1}(\overline{ba}). \end{aligned}$$

Lemma 2.4.3. (1) Any non-zero element $x \in \mathfrak{C}^C$ can be transformed to 1 by successive actions of \mathfrak{d}_4^C .

(2) \mathfrak{C}^C is a \mathfrak{d}_4^C -irreducible C -module.

$$(3) \quad \mathfrak{d}_4^C \mathfrak{C}^C = \left\{ \sum_i D_i a_i \mid D_i \in \mathfrak{d}_4^C, a_i \in \mathfrak{C}^C \right\} = \mathfrak{C}^C.$$

Proof. (1) Let $0 \neq x = \sum_{i=0}^7 x_i e_i$, $x_i \in C$. Suppose $x_i \neq 0$ (moreover we may assume $i \neq 0$). Then we have

$$x \xrightarrow{G_{i0}} x_i e_0 - x_0 e_i \xrightarrow{G_{j0}(j \neq i)} x_i e_j \xrightarrow{x_j^{-1} G_{0j}} e_0 = 1.$$

The case $x_0 \neq 0$ is reduced to the above by the action of G_{i0} .

(2) Let $W \neq \{0\}$ be a \mathfrak{d}_4^C -invariant C -submodule of \mathfrak{C}^C . If $e_0 = 1 \in W$, then $G_{i0}e_0 = e_i$. Therefore $W = \mathfrak{C}^C$. Now, since a non-zero element x of W can be transformed to 1 by \mathfrak{d}_4^C from (1) above, W contains 1, so that $W = \mathfrak{C}^C$, because $G_{i0}e_0 = e_i$.

(3) Since $\mathfrak{d}_4^C \mathfrak{C}^C$ is a \mathfrak{d}_4^C -invariant C -submodule of \mathfrak{C}^C , from the irreducibility of \mathfrak{C}^C of (2), we have $\mathfrak{d}_4^C \mathfrak{C}^C = \mathfrak{C}^C$.

Recall that any element $\delta \in \mathfrak{f}_4^C$ is uniquely expressed by

$$\delta = D + \tilde{A}_1(a_1) + \tilde{A}_2(a_2) + \tilde{A}_3(a_3), \quad D \in \mathfrak{d}_4^C, a_i \in \mathfrak{C}^C,$$

where $A_i(a_i) \in (\mathfrak{M}^-)^C$ (Theorem 2.3.8).

Theorem 2.4.4. *The Lie algebra \mathfrak{f}_4^C is simple and so \mathfrak{f}_4 is also simple.*

Proof. Let denote $\tilde{\mathfrak{A}}_i^C = \{\tilde{A}_i(a) \mid a \in \mathfrak{C}^C\}$ and $\tilde{\mathfrak{A}}^C = \tilde{\mathfrak{A}}_1^C \oplus \tilde{\mathfrak{A}}_2^C \oplus \tilde{\mathfrak{A}}_3^C$, then

$$\mathfrak{f}_4^C = \mathfrak{d}_4^C \oplus \tilde{\mathfrak{A}}_1^C \oplus \tilde{\mathfrak{A}}_2^C \oplus \tilde{\mathfrak{A}}_3^C = \mathfrak{d}_4^C \oplus \tilde{\mathfrak{A}}^C.$$

Let $p : \mathfrak{f}_4^C \rightarrow \mathfrak{d}_4^C$ and $q : \mathfrak{f}_4^C \rightarrow \tilde{\mathfrak{A}}^C$ be projections of $\mathfrak{f}_4^C = \mathfrak{d}_4^C \oplus \tilde{\mathfrak{A}}^C$. Now, let \mathfrak{a} be a non-zero ideal of \mathfrak{f}_4^C . Then $p(\mathfrak{a})$ is an ideal of \mathfrak{d}_4^C . Indeed, if $D \in p(\mathfrak{a})$, then there exist $a_i \in \mathfrak{C}^C$, $i = 1, 2, 3$, such that $D + \sum_{i=1}^3 \tilde{A}_i(a_i) \in \mathfrak{a}$. For any $D' \in \mathfrak{d}_4^C$, we have

$$\mathfrak{a} \ni \left[D', D + \sum_{i=1}^3 \tilde{A}_i(a_i) \right] = [D', D] + \sum_{i=1}^3 \tilde{A}_i(D'_i a_i) \quad (\text{Lemma 2.4.2}),$$

hence $[D', D] \in p(\mathfrak{a})$.

We show that either $\mathfrak{d}_4^C \cap \mathfrak{a} \neq \{0\}$ or $\tilde{\mathfrak{A}}^C \cap \mathfrak{a} \neq \{0\}$. Assume that $\mathfrak{d}_4^C \cap \mathfrak{a} = \{0\}$ and $\tilde{\mathfrak{A}}^C \cap \mathfrak{a} = \{0\}$. The mapping $p|_{\mathfrak{a}} : \mathfrak{a} \rightarrow \mathfrak{d}_4^C$ is injective because $\tilde{\mathfrak{A}}^C \cap \mathfrak{a} = \{0\}$. Since $p(\mathfrak{a})$ is a non-zero ideal of \mathfrak{d}_4^C and \mathfrak{d}_4^C is simple, we have $p(\mathfrak{a}) = \mathfrak{d}_4^C$. Hence $\dim_C \mathfrak{a} = \dim_C p(\mathfrak{a}) = \dim_C \mathfrak{d}_4^C = 28$. On the other hand, since $\mathfrak{d}_4^C \cap \mathfrak{a} = \{0\}$, $q|_{\mathfrak{a}} : \mathfrak{a} \rightarrow \tilde{\mathfrak{A}}^C$ is also injective, we have $\dim_C \mathfrak{a} \leq \dim_C \tilde{\mathfrak{A}}^C = 8 \times 3 = 24$. This leads to a contradiction.

We now consider the following two cases.

(1) Case $\mathfrak{d}_4^C \cap \mathfrak{a} \neq \{0\}$. From the simplicity of \mathfrak{d}_4^C , we have $\mathfrak{d}_4^C \cap \mathfrak{a} = \mathfrak{d}_4^C$, hence $\mathfrak{a} \supset \mathfrak{d}_4^C$. On the other hand, we have

$$\mathfrak{a} \supset [\mathfrak{a}, \mathfrak{f}_4^C] \supset [\mathfrak{d}_4^C, \tilde{\mathfrak{A}}_i^C] = \tilde{\mathfrak{A}}_i^C, \quad i = 1, 2, 3.$$

The last equation follows from $[D, \tilde{A}_i(a_i)] = \tilde{A}_i(Da_i)$ (Lemma 2.4.2) and $\mathfrak{d}_4^C \mathfrak{C}^C = \mathfrak{C}^C$ (Lemma 2.4.3.(3)). Hence $\mathfrak{a} \supset \mathfrak{d}_4^C \oplus \tilde{\mathfrak{A}}_1^C \oplus \tilde{\mathfrak{A}}_2^C \oplus \tilde{\mathfrak{A}}_3^C = \mathfrak{f}_4^C$.

(2) Case $\tilde{\mathfrak{A}}^C \cap \mathfrak{a} \neq \{0\}$. Choose a non-zero element

$$\tilde{A}_1(a_1) + \tilde{A}_2(a_2) + \tilde{A}_3(a_3) \in \tilde{\mathfrak{A}}^C \cap \mathfrak{a} \subset \mathfrak{a}.$$

If $a_1 \neq 0$, under the actions of \mathfrak{d}_4^C of Lemma 2.4.3.(1), we have

$$\tilde{A}_1(1) + \tilde{A}_2(b) + \tilde{A}_3(c) \in \mathfrak{a}.$$

If $b = c = 0$, then $\tilde{A}_1(1) \in \mathfrak{a}$, hence from Lemma 2.4.2,

$$0 \neq [\tilde{A}_1(1), \tilde{A}_1(e_1)] \in \mathfrak{d}_4^C \cap \mathfrak{a}$$

(as for the first inequality, note that $[\tilde{A}_1(1), \tilde{A}_1(e_1)]F_2(1) = -2F_2(e_1)$), so the case is reduced to the case (1). If $b \neq 0$, then take the Lie bracket with $\tilde{A}_1(1)$, then

$$-\tilde{A}_3(\bar{b}) + \tilde{A}_2(\bar{c}) \in \mathfrak{a} \quad \text{and then} \quad \tilde{A}_3(1) + \tilde{A}_2(c') \in \mathfrak{a}$$

under some actions of \mathfrak{d}_4^C . If $c' = 0$, then this can be reduced to the case (1) as in the above. The case $c' \neq 0$ can also be reduced to the case (1) by the same argument as above. Thus we have $\mathfrak{a} = \mathfrak{f}_4^C$, which proves the simplicity of \mathfrak{f}_4^C .

Lemma 2.4.5. (1) For $\delta \in \mathfrak{f}_4^C$ and $A, B \in \mathfrak{J}^C$, we have

$$[\delta, [\tilde{A}, \tilde{B}]] = [\delta\tilde{A}, \tilde{B}] + [\tilde{A}, \delta\tilde{B}].$$

(2) Any element $\delta \in \mathfrak{f}_4^C$ is expressed as $\delta = \sum_i [\tilde{A}_i, \tilde{B}_i]$, $A_i, B_i \in \mathfrak{J}^C$.

Proof. (1) $[\delta, [\tilde{A}, \tilde{B}]]X = \delta[\tilde{A}, \tilde{B}]X - [\tilde{A}, \tilde{B}]\delta X$
 $= \delta(A \circ (B \circ X) - B \circ (A \circ X)) - (A \circ (B \circ \delta X) - B \circ (A \circ \delta X))$
 $= \delta A \circ (B \circ X) + A \circ (\delta B \circ X) + A \circ (B \circ \delta X) - \delta B \circ (A \circ X)$
 $\quad - B \circ (\delta A \circ X) - B \circ (A \circ \delta X) - A \circ (B \circ \delta X) + B \circ (A \circ \delta X)$
 $= \delta A \circ (B \circ X) - B \circ (\delta A \circ X) + A \circ (\delta B \circ X) - \delta B \circ (A \circ X)$
 $= [\delta\tilde{A}, \tilde{B}]X + [\tilde{A}, \delta\tilde{B}]X, \quad X \in \mathfrak{J}^C.$

(2) By (1) we see that $\mathfrak{a} = \{\sum_i [\tilde{A}_i, \tilde{B}_i] \mid A_i, B_i \in \mathfrak{J}^C\}$ is an ideal of \mathfrak{f}_4^C . From the simplicity of \mathfrak{f}_4^C , we have $\mathfrak{a} = \mathfrak{f}_4^C$.

Proposition 2.4.6. (1) $\mathfrak{J}_0^C = \{X \in \mathfrak{J}^C \mid \text{tr}(X) = 0\}$ is an \mathfrak{f}_4^C -irreducible C -module.

(2) $\mathfrak{f}_4^C \mathfrak{J}_0^C = \left\{ \sum_i \delta_i B_i \mid \delta_i \in \mathfrak{f}_4^C, B_i \in \mathfrak{J}_0^C \right\} = \mathfrak{J}_0^C$.

Proof. (1) Evidently \mathfrak{J}_0^C is an \mathfrak{f}_4^C - C -module (Lemma 2.2.1.(2)). Now, let W be a non-zero \mathfrak{f}_4^C -invariant C -submodule of \mathfrak{J}_0^C .

Case (i) If W contains some $F_i(x) \neq 0$ ($i = 1, 2, 3$), then $W = \mathfrak{J}_0^C$. Indeed, under some actions of \mathfrak{d}_4^C , we can assume $F_i(1) \in W$ (Lemma 2.4.3.(1)). Next, from

$$F_i(1) \xrightarrow{\tilde{A}_i(1)} E_{i+1} - E_{i+2} \xrightarrow{\tilde{A}_{i+2}(1)} F_{i+2}(1) \longrightarrow \cdots \longrightarrow F_{i+1}(1),$$

and $\tilde{A}_i(a)F_{i+2}(1) = F_{i+1}(\bar{a})$, etc., we have $E_i - E_{i+1}, F_i(a) \in W$ and hence we have $W = \mathfrak{J}_0^C$. Now, in the general case, choose a non-zero element X from W :

$$0 \neq X = \xi(E_1 - E_2) + \eta(E_2 - E_3) + F_1(x_1) + F_2(x_2) + F_3(x_3) \in W.$$

Case $x_1 \neq 0$. Under the action of \mathfrak{d}_4^C of Lemma 2.4.3.(1), we have

$$F_1(1) + F_2(y_2) + F_3(y_3) \in W.$$

If $y_2 = y_3 = 0$, then $F_1(1) \in W$, hence we can reduce to the case (i). If $y_2 \neq 0$, choose $a \in \mathfrak{C}^C$, $a \neq 0$, such that $(y_2, a) = 0$, and apply $\tilde{A}_2(a)$, then we have $-F_3(\bar{a}) + F_1(\overline{ay_3}) \in W$. Again under the action of \mathfrak{d}_4^C , we have

$$F_3(1) + F_1(z_1) \in W.$$

If $z_1 = 0$, we can reduce to the case (i). If $z_1 \neq 0$, choose $b \in \mathfrak{C}^C$, $b \neq 0$, such that $(z_1, b) = 0$ and apply $\tilde{A}_1(b)$, then $F_2(\bar{b}) \in W$, which is again reduced to the case (i). Similarly for $x_2 \neq 0$ or $x_3 \neq 0$ the arguments above hold.

Case $x_1 = x_2 = x_3 = 0$. Applying $\tilde{A}_1(1)$ on the non-zero element

$$X = \xi(E_1 - E_2) + \eta(E_2 - E_3) \in W,$$

we have $F_1(\xi - 2\eta) \in W$. If $\xi - 2\eta \neq 0$, we can reduce to the case (i). If $\xi - 2\eta = 0$, applying $\tilde{A}_3(1)$, we have $F_3(3\eta) \in W$ which is again the case (i). We have just proved that $W = \mathfrak{J}_0^C$.

(2) Since $\mathfrak{f}_4^C \mathfrak{J}_0^C$ is an \mathfrak{f}_4^C -invariant C -submodule of \mathfrak{J}_0^C , we have $\mathfrak{f}_4^C \mathfrak{J}_0^C = \mathfrak{J}_0^C$ from (1).

2.5. Killing form of \mathfrak{f}_4^C

Lemma 2.5.1. *For $A, B, C, D \in \mathfrak{J}^C$, we have*

$$([\tilde{A}, \tilde{B}]C, D) = ([\tilde{C}, \tilde{D}]A, B).$$

Proof.
$$\begin{aligned} ([\tilde{A}, \tilde{B}]C, D) &= (A \circ (B \circ C), D) - (B \circ (A \circ C), D) \\ &= (B \circ C, A \circ D) - (A \circ C, B \circ D) \\ &= (C \circ (D \circ A), B) - (D \circ (C \circ A), B) = ([\tilde{C}, \tilde{D}]A, B). \end{aligned}$$

Definition. In \mathfrak{f}_4^C , we define an inner product $(\delta_1, \delta_2)_4$ by

$$(\delta, [\tilde{A}, \tilde{B}])_4 = (\delta A, B), \quad \delta \in \mathfrak{f}_4^C, \quad A, B \in \mathfrak{J}^C.$$

More precisely, we define

$$(\delta_1, \delta_2)_4 = \sum_{i,j} ([\tilde{A}_i, \tilde{B}_i]C_j, D_j) = \sum_{i,j} ([\tilde{C}_j, \tilde{D}_j]A_i, B_i),$$

for $\delta_1 = \sum_i [\tilde{A}_i, \tilde{B}_i]$, $\delta_2 = \sum_j [\tilde{C}_j, \tilde{D}_j]$, $A_i, B_i, C_j, D_j \in \mathfrak{J}^C$ (Lemma 2.4.5.(2)). Then, Lemma 2.5.1 shows that the definition of the inner product $(\delta_1, \delta_2)_4$ is independent of the choice of the expressions of δ_1, δ_2 , and that $(\delta_1, \delta_2)_4$ is symmetric.

Lemma 2.5.2. *The inner product $(\delta_1, \delta_2)_4$ of \mathfrak{f}_4^C is \mathfrak{f}_4^C -adjoint invariant:*

$$([\delta, \delta_1], \delta_2)_4 + (\delta_1, [\delta, \delta_2])_4 = 0, \quad \delta, \delta_i \in \mathfrak{f}_4^C.$$

Proof. It is sufficient to show the lemma for $\delta_1 = [\tilde{A}, \tilde{B}]$, $\delta_2 = [\tilde{C}, \tilde{D}]$, $A, B, C, D \in \mathfrak{J}$.

$$\begin{aligned} & ([\delta, [\tilde{A}, \tilde{B}]], [\tilde{C}, \tilde{D}])_4 + ([\tilde{A}, \tilde{B}], [\delta, [\tilde{C}, \tilde{D}]])_4 \\ &= ([\delta\tilde{A}, \tilde{B}] + [\tilde{A}, \delta\tilde{B}], [\tilde{C}, \tilde{D}])_4 + ([\tilde{A}, \tilde{B}], [\delta\tilde{C}, \tilde{D}] + [\tilde{C}, \delta\tilde{D}])_4 \quad (\text{Lemma 2.4.5.(1)}) \\ &= ([\delta\tilde{A}, \tilde{B}]C, D) + ([\tilde{A}, \delta\tilde{B}]C, D) + ([\tilde{A}, \tilde{B}]\delta C, D) + ([\tilde{A}, \tilde{B}]C, \delta D) \\ &= (\delta A \circ (B \circ C), D) - (B \circ (\delta A \circ C), D) + (A \circ (\delta B \circ C), D) - (\delta B \circ (A \circ C), D) \\ &\quad + (A \circ (B \circ \delta C), D) - (B \circ (A \circ \delta C), D) + (A \circ (B \circ C), \delta D) - (B \circ (A \circ C), \delta D), \end{aligned}$$

which is equal to 0, if we use the relation $(X \circ Y, \delta Z) = -(\delta X \circ Y, Z) - (X \circ \delta Y, Z)$ for the above last two terms.

Theorem 2.5.3. *The Killing form B_4 of the Lie algebra \mathfrak{f}_4^C is given by*

$$B_4(\delta_1, \delta_2) = 9(\delta_1, \delta_2)_4 = 3\text{tr}(\delta_1\delta_2), \quad \delta_1, \delta_2 \in \mathfrak{f}_4^C.$$

Proof. Since \mathfrak{f}_4^C is simple (Theorem 2.4.4), there exist $k, k' \in C$ such that

$$B_4(\delta_1, \delta_2) = k(\delta_1, \delta_2)_4 = k'\text{tr}(\delta_1\delta_2).$$

To determine these k, k' , we put $\delta = \delta_1 = \delta_2 = \tilde{A}_1(1)$. Since $\tilde{A}_1(1) = -2[\tilde{E}_3, \tilde{F}_1(1)]$ (Lemma 2.4.2), we have

$$\begin{aligned} (\delta, \delta)_4 &= (\tilde{A}_1(1), \tilde{A}_1(1))_4 = -2(\tilde{A}_1(1), [\tilde{E}_3, \tilde{F}_1(1)])_4 \\ &= -2(\tilde{A}_1(1)E_3, F_1(1)) = -(F_1(1), F_1(1)) = -2. \end{aligned}$$

On the other hand, $(\text{ad}\delta)^2$ is calculated as follows.

$$\begin{aligned} [\tilde{A}_1(1), [\tilde{A}_1(1), G_{i0}]] &= -[\tilde{A}_1(1), \tilde{A}_1(e_i)] = -G_{i0}, \quad i \neq 0 \\ [\tilde{A}_1(1), [\tilde{A}_1(1), \tilde{A}_1(e_i)]] &= [\tilde{A}_1(1), G_{i0}] = -\tilde{A}_1(e_i), \quad i \neq 0 \\ [\tilde{A}_1(1), [\tilde{A}_1(1), \tilde{A}_2(e_i)]] &= \frac{1}{2}[\tilde{A}_1(1), \tilde{A}_3(e_i)] = -\frac{1}{4}\tilde{A}_2(e_i), \\ [\tilde{A}_1(1), [\tilde{A}_1(1), \tilde{A}_3(e_i)]] &= -\frac{1}{2}[\tilde{A}_1(1), \tilde{A}_2(e_i)] = -\frac{1}{4}\tilde{A}_3(e_i), \\ \text{the others} &= 0. \end{aligned}$$

Hence we have

$$B_4(\delta, \delta) = \text{tr}((\text{ad}\tilde{A}_1(1))^2) = (-1) \times 7 \times 2 + \left(-\frac{1}{4}\right) \times 8 \times 2 = -18.$$

Therefore $k = 9$. Next, we will calculate $\text{tr}(\delta\delta)$.

$$\begin{aligned}
\tilde{A}_1(1)\tilde{A}_1(1)E_2 &= -\frac{1}{2}\tilde{A}_1(1)F_1(1) = -\frac{1}{2}(E_2 - E_3), \\
\tilde{A}_1(1)\tilde{A}_1(1)E_3 &= \frac{1}{2}\tilde{A}_1(1)F_1(1) = \frac{1}{2}(E_2 - E_3), \\
\tilde{A}_1(1)\tilde{A}_1(1)F_1(1) &= \tilde{A}_1(1)(E_2 - E_3) = -F_1(1), \\
\tilde{A}_1(1)\tilde{A}_1(1)F_2(e_i) &= \frac{1}{2}\tilde{A}_1(1)F_3(\overline{e_i}) = -\frac{1}{4}F_2(e_i), \\
\tilde{A}_1(1)\tilde{A}_1(1)F_3(e_i) &= -\frac{1}{2}\tilde{A}_1(1)F_3(\overline{e_i}) = -\frac{1}{4}F_3(e_i), \\
\text{the others} &= 0.
\end{aligned}$$

Hence we have

$$\text{tr}(\delta\delta) = \text{tr}((\tilde{A}_1(1))^2) = \left(-\frac{1}{2}\right) \times 2 - 1 + \left(-\frac{1}{4}\right) \times 8 \times 2 = -6.$$

Therefore $k' = 3$.

Lemma 2.5.4. *For a non-zero element $A \in \mathfrak{J}_0^C$, there exists $B \in \mathfrak{J}_0^C$ such that $[\tilde{A}, \tilde{B}] \neq 0$.*

Proof. Assume that $[\tilde{A}, \tilde{B}] = 0$ for all $B \in \mathfrak{J}_0^C$. Then $0 = (\delta, [\tilde{A}, \tilde{B}])_4 = -(\delta, [\tilde{B}, \tilde{A}])_4 = -(\delta B, A)$ for any $\delta \in \mathfrak{f}_4^C$. Since $\mathfrak{f}_4^C \mathfrak{J}_0^C = \mathfrak{J}_0^C$ (Proposition 2.4.6.(2)), we have $(\mathfrak{J}_0^C, A) = 0$, so that $A = 0$.

2.6. Roots of \mathfrak{f}_4^C

Before we obtain the roots of the Lie algebra \mathfrak{f}_4^C , we recall the roots of the Lie algebra \mathfrak{D}_4^C :

$$\mathfrak{D}_4^C = \{D \in \text{Hom}_C(\mathfrak{C}^C) \mid (Dx, y) + (x, Dy) = 0\}.$$

If we let $H_k = -iG_{k4+k}$ for $k = 0, 1, 2, 3$, then

$$\mathfrak{h} = \{H = \sum_{k=0}^3 \lambda_k H_k \mid \lambda_k \in C\}$$

is a Cartan subalgebra of \mathfrak{D}_4^C , and roots of \mathfrak{D}_4^C relative to \mathfrak{h} are given by

$$\pm(\lambda_k - \lambda_l), \quad \pm(\lambda_k + \lambda_l), \quad 0 \leq k < l \leq 3.$$

The root vectors associated with these roots are respectively given by

$$\begin{aligned}
\lambda_k - \lambda_l &: (G_{kl} + G_{4+k4+l}) - i(G_{k4+l} + G_{l4+k}), \\
-\lambda_k + \lambda_l &: i(G_{kl} + G_{4+k4+l}) - (G_{k4+l} + G_{l4+k}), \\
\lambda_k + \lambda_l &: (G_{kl} - G_{4+k4+l}) + i(G_{k4+l} - G_{l4+k}), \\
-\lambda_k - \lambda_l &: i(G_{kl} - G_{4+k4+l}) + (G_{k4+l} - G_{l4+k}).
\end{aligned}$$

The Lie algebra \mathfrak{D}_4^C is contained in \mathfrak{f}_4^C as

$$\mathfrak{D}_4^C \ni D \rightarrow (D, \nu D, \kappa \pi D) \in \mathfrak{d}_4^C \subset \mathfrak{f}_4^C$$

(Theorem 1.3.6, Proposition 2.3.7). Outer automorphisms ν, π of \mathfrak{D}_4^C induce C -linear transformations of \mathfrak{h} and the matrices of ν, π with respect to the C -basis H_0, H_1, H_2, H_3 of \mathfrak{h} are respectively given by

$$\nu = \pi \kappa = \frac{1}{2} \begin{pmatrix} -1 & -1 & 1 & -1 \\ 1 & 1 & 1 & -1 \\ -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \end{pmatrix}, \quad \pi = \frac{1}{2} \begin{pmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 \end{pmatrix}$$

(Lemma 1.3.1). Note that they are orthogonal matrices.

Theorem 2.6.1. *The rank of the Lie algebra \mathfrak{f}_4^C is 4. The roots of \mathfrak{f}_4^C relative to some Cartan subalgebra of \mathfrak{f}_4^C are given by*

$$\begin{aligned} & \pm(\lambda_k - \lambda_l), \quad \pm(\lambda_k + \lambda_l), \quad 0 \leq k < l \leq 3, \\ & \pm\lambda_0, \quad \pm\lambda_1, \quad \pm\lambda_2, \quad \pm\lambda_3, \\ & \pm\frac{1}{2}(-\lambda_0 - \lambda_1 + \lambda_2 - \lambda_3), \quad \pm\frac{1}{2}(\lambda_0 + \lambda_1 + \lambda_2 - \lambda_3), \\ & \pm\frac{1}{2}(-\lambda_0 + \lambda_1 + \lambda_2 + \lambda_3), \quad \pm\frac{1}{2}(\lambda_0 - \lambda_1 + \lambda_2 + \lambda_3), \\ & \pm\frac{1}{2}(\lambda_0 - \lambda_1 + \lambda_2 - \lambda_3), \quad \pm\frac{1}{2}(-\lambda_0 + \lambda_1 + \lambda_2 - \lambda_3), \\ & \pm\frac{1}{2}(\lambda_0 + \lambda_1 + \lambda_2 + \lambda_3), \quad \pm\frac{1}{2}(-\lambda_0 - \lambda_1 + \lambda_2 + \lambda_3). \end{aligned}$$

Proof. We use the decomposition in Theorem 2.4.4:

$$\mathfrak{f}_4^C = \mathfrak{d}_4^C \oplus \tilde{\mathfrak{A}}_1^C \oplus \tilde{\mathfrak{A}}_2^C \oplus \tilde{\mathfrak{A}}_3^C.$$

Let $\mathfrak{h} = \left\{ H = \sum_{k=0}^3 \lambda_k H_k \mid \lambda_k \in C \right\} \subset \mathfrak{d}_4^C \subset \mathfrak{f}_4^C$. Since \mathfrak{h} is a Cartan subalgebra of \mathfrak{d}_4^C (it will be also a Cartan subalgebra of \mathfrak{f}_4^C), the roots of \mathfrak{d}_4^C :

$$\pm(\lambda_k - \lambda_l), \quad \pm(\lambda_k + \lambda_l), \quad 0 \leq k < l \leq 3$$

are also roots of \mathfrak{f}_4^C . Furthermore, from $[H, \tilde{A}_1(a)] = \tilde{A}_1(Ha)$ (Lemma 2.4.2), where

$$\begin{aligned} H(e_k + ie_{4+k}) &= - \sum_{j=0}^3 \lambda_k i G_{j4+j}(e_k + ie_{4+k}) \\ &= -i\lambda_k(-e_{4+k} + ie_k) = \lambda_k(e_k + ie_{4+k}), \end{aligned}$$

we see that λ_k is a root of \mathfrak{f}_4^C and $\tilde{A}_1(e_k + ie_{4+k})$ is an associated root vector for $0 \leq k \leq 3$. Similarly, $-\lambda_k$ for $0 \leq k \leq 3$ is root of \mathfrak{f}_4^C and $\tilde{A}_1(e_k - ie_{4+k})$ is

an associated root vector. Next, by Lemma 2.4.2 we have $[H, \tilde{A}_2(a)] = \tilde{A}_2((\nu H)a)$, where

$$\begin{aligned}\nu H &= \nu \left(\sum_{k=0}^3 \lambda_k H_k \right) \\ &= \frac{1}{2}(\lambda_0(-H_0 + H_1 - H_2 + H_3) + \lambda_1(-H_0 + H_1 + H_2 - H_3) \\ &\quad + \lambda_2(H_0 + H_1 + H_2 + H_3) + \lambda_3(-H_0 - H_1 + H_2 + H_3)) \\ &= \frac{1}{2}(-\lambda_0 - \lambda_1 + \lambda_2 - \lambda_3)H_0 + \frac{1}{2}(\lambda_0 + \lambda_1 + \lambda_2 - \lambda_3)H_1 \\ &\quad + \frac{1}{2}(-\lambda_0 + \lambda_1 + \lambda_2 + \lambda_3)H_2 + \frac{1}{2}(\lambda_0 - \lambda_1 + \lambda_2 + \lambda_3)H_3,\end{aligned}$$

and so we see that coefficients of H_0, H_1, H_2, H_3 are roots of \mathfrak{f}_4^C and that $\tilde{A}_2(e_k + i + ie_{4+k})$ is associated root vector for $0 \leq k \leq 3$. The roots above with negative sign are also roots of \mathfrak{f}_4^C and $\tilde{A}_2(e_k - ie_{4+k})$ are associated root vectors. Finally, from $[H, \tilde{A}_3(a)] = \tilde{A}_3((\kappa\pi H)a)$ (Lemma 2.4.2), where

$$\begin{aligned}\kappa\pi H &= \frac{1}{2}(-\lambda_0 + \lambda_1 - \lambda_2 + \lambda_3)H_0 + \frac{1}{2}(-\lambda_0 + \lambda_1 + \lambda_2 - \lambda_3)H_1 \\ &\quad + \frac{1}{2}(\lambda_0 + \lambda_1 + \lambda_2 + \lambda_3)H_2 + \frac{1}{2}(-\lambda_0 - \lambda_1 + \lambda_2 + \lambda_3)H_3,\end{aligned}$$

we obtain the remainders of roots.

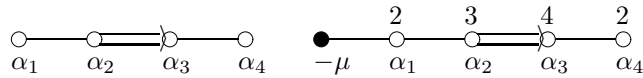
Theorem 2.6.2. *In the root system of Theorem 2.6.1,*

$$\alpha_1 = \lambda_0 - \lambda_1, \quad \alpha_2 = \lambda_1 - \lambda_2, \quad \alpha_3 = \lambda_2, \quad \alpha_4 = \frac{1}{2}(-\lambda_0 - \lambda_1 - \lambda_2 + \lambda_3)$$

is a fundamental root system of the Lie algebra \mathfrak{f}_4^C and

$$\mu = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4$$

is the highest root. The Dynkin diagram and the extended Dynkin diagram of \mathfrak{f}_4^C are respectively given by



Proof. All positive roots of \mathfrak{f}_4^C are expressed by $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ as follows.

$$\begin{aligned}\lambda_0 &= \alpha_1 + \alpha_2 + \alpha_3 \\ \lambda_1 &= \alpha_2 + \alpha_3 \\ \lambda_3 &= \alpha_3 \\ \lambda_3 &= \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4,\end{aligned}$$

$$\begin{aligned}
\lambda_0 - \lambda_1 &= \alpha_1 \\
\lambda_0 - \lambda_2 &= \alpha_1 + \alpha_2 \\
-\lambda_0 + \lambda_3 &= \alpha_2 + 2\alpha_3 + 2\alpha_4 \\
\lambda_1 - \lambda_2 &= \alpha_2 \\
-\lambda_1 + \lambda_3 &= \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 \\
-\lambda_2 + \lambda_3 &= \alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4 \\
\lambda_0 + \lambda_1 &= \alpha_1 + 2\alpha_2 + 2\alpha_3 \\
\lambda_0 + \lambda_2 &= \alpha_1 + \alpha_2 + 2\alpha_3 \\
\lambda_0 + \lambda_3 &= 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4 \\
\lambda_1 + \lambda_2 &= \alpha_2 + 2\alpha_3 \\
\lambda_1 + \lambda_3 &= \alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4 \\
\lambda_2 + \lambda_3 &= \alpha_1 + 2\alpha_2 + 4\alpha_3 + 4\alpha_4, \\
\frac{1}{2}(\lambda_0 + \lambda_1 + \lambda_2 + \lambda_3) &= \alpha_1 + 2\alpha_2 + 3\alpha_3 + \alpha_4 \\
\frac{1}{2}(-\lambda_0 - \lambda_1 - \lambda_2 + \lambda_3) &= \alpha_4 \\
\frac{1}{2}(\lambda_0 + \lambda_1 - \lambda_2 + \lambda_3) &= \alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4 \\
\frac{1}{2}(\lambda_0 - \lambda_1 + \lambda_2 + \lambda_3) &= \alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4 \\
\frac{1}{2}(-\lambda_0 - \lambda_1 + \lambda_2 + \lambda_3) &= \alpha_3 + \alpha_4 \\
\frac{1}{2}(-\lambda_0 + \lambda_1 - \lambda_2 + \lambda_3) &= \alpha_2 + \alpha_3 + \alpha_4 \\
\frac{1}{2}(\lambda_0 - \lambda_1 - \lambda_2 + \lambda_3) &= \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 \\
\frac{1}{2}(-\lambda_0 + \lambda_1 + \lambda_2 + \lambda_3) &= \alpha_2 + 2\alpha_3 + \alpha_4.
\end{aligned}$$

Hence $\Pi = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ is a fundamental root system of \mathfrak{f}_4^C . The real part $\mathfrak{h}_{\mathbf{R}}$ of \mathfrak{h} is

$$\mathfrak{h}_{\mathbf{R}} = \{H = \sum_{k=0}^3 \lambda_k H_k \mid \lambda_k \in \mathbf{R}\}.$$

The Killing form B_4 of \mathfrak{f}_4^C is $B_4(\delta_1, \delta_2) = 3\text{tr}(\delta_1 \delta_2)$ (Theorem 2.5.3), so that

$$B_4(H, H') = 18 \sum_{k=0}^3 \lambda_k \lambda_k', \quad H = \sum_{k=0}^3 \lambda_k H_k, H' = \sum_{k=0}^3 \lambda_k' H_k \in \mathfrak{h}_{\mathbf{R}}.$$

Indeed, since

$$\begin{aligned}
HE_i &= 0, \quad i = 1, 2, 3, \\
HF_1(x) &= F_1(Hx), \quad HF_2(x) = F_2((\nu H)x), \quad HF_3(x) = F_3((\kappa\pi H)x),
\end{aligned}$$

we have

$$\begin{aligned}
B_4(H, H') &= \sum_{k,l=0}^3 \lambda_k \lambda_l' B_4(H_k, H_l) \\
&= 3 \sum_{k,l} (\lambda_k \lambda_l' (\text{tr}(H_k H_l) + \text{tr}((\nu H_k)(\nu H_l)) + \text{tr}((\kappa \pi H_k)(\kappa \pi H_l)))) \\
&= 3 \sum_{k,l} \lambda_k \lambda_l' (2 + 2 + 2) \delta_{kl} = 18 \sum_{k=0}^3 \lambda_k \lambda_k'.
\end{aligned}$$

Now, the canonical elements $H_{\alpha_i} \in \mathfrak{h}_{\mathbf{R}}$ corresponding to α_i ($B_4(H_{\alpha_i}, H) = \alpha(H)$, $H \in \mathfrak{h}_{\mathbf{R}}$) are determined by

$$\begin{aligned}
H_{\alpha_1} &= \frac{1}{18}(H_0 - H_1), & H_{\alpha_2} &= \frac{1}{18}(H_1 - H_2), \\
H_{\alpha_3} &= \frac{1}{18}H_2, & H_{\alpha_4} &= \frac{1}{36}(-H_0 - H_1 - H_2 + H_3).
\end{aligned}$$

Hence we have

$$(\alpha_1, \alpha_1) = B_4(H_{\alpha_1}, H_{\alpha_1}) = 18 \frac{1}{18} \frac{1}{18} (1 + 1) = \frac{1}{9}$$

and the other inner products are similarly calculated. Hence, the inner products induced by the Killing form B_4 between $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ and $-\mu$ are given by

$$\begin{aligned}
(\alpha_1, \alpha_1) &= (\alpha_2, \alpha_2) = \frac{1}{9}, & (\alpha_3, \alpha_3) &= (\alpha_4, \alpha_4) = \frac{1}{18}, \\
(\alpha_1, \alpha_2) &= -\frac{1}{18}, & (\alpha_2, \alpha_3) &= -\frac{1}{18}, & (\alpha_3, \alpha_4) &= -\frac{1}{36}, \\
(\alpha_1, \alpha_3) &= (\alpha_1, \alpha_4) = (\alpha_2, \alpha_4) &= 0, \\
(-\mu, -\mu) &= \frac{1}{9}, & (-\mu, \alpha_1) &= -\frac{1}{18}, & (-\mu, \alpha_i) &= 0, \quad i = 2, 3, 4,
\end{aligned}$$

using them, we can draw the Dynkin diagram and the extended Dynkin diagram of \mathfrak{f}_4^C .

According Borel-Siebert theory, the Lie algebra \mathfrak{f}_4 has three subalgebras as maximal subalgebras with the maximal rank 4.

(1) The first one is a subalgebra of type B_4 obtained as the fixed points of an involution σ of \mathfrak{f}_4 .

(2) The second one is a subalgebra of type $C_1 \oplus C_3$ obtained as the fixed points of an involution γ of \mathfrak{f}_4 .

(3) The third one is a subalgebra of type $A_2 \oplus A_2$ obtained as the fixed points of an automorphism w of order 3 of \mathfrak{f}_4 .

These subalgebras will be realized as subgroups of the group F_4 in Theorems 2.9.1, 2.11.2 and 2.12.2, respectively.

2.7. Subgroup $Spin(9)$ of F_4

Theorem 2.7.1. $\{\alpha \in F_4 \mid \alpha E_i = E_i, i = 1, 2, 3\} \cong Spin(8)$.

(From now on, we identify these groups).

Proof. We recall the group $Spin(8)$ of Theorem 1.16.2 and define a mapping $\varphi : Spin(8) \rightarrow D_4 = \{\alpha \in F_4 \mid \alpha E_i = E_i, i = 1, 2, 3\}$ by

$$\varphi(\alpha_1, \alpha_2, \alpha_3) \begin{pmatrix} \xi_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & \xi_2 & x_1 \\ x_2 & \bar{x}_1 & \xi_3 \end{pmatrix} = \begin{pmatrix} \xi_1 & \alpha_3 x_3 & \overline{\alpha_2 x_2} \\ \overline{\alpha_3 x_3} & \xi_2 & \alpha_1 x_1 \\ \alpha_2 x_2 & \overline{\alpha_1 x_1} & \xi_3 \end{pmatrix}.$$

We first prove that $\alpha = \varphi(\alpha_1, \alpha_2, \alpha_3) \in D_4$. Indeed, the fact that $\det(\alpha X) = \det X$ can be seen by observing that

$$\begin{aligned} R((\alpha_1 x_1)(\alpha_2 x_2)(\alpha_3 x_3)) &= R(\overline{(\alpha_3(\bar{x}_1 \bar{x}_2))} \alpha_3 x_3) \\ &= (\alpha_3(\bar{x}_1 \bar{x}_2), \alpha_3 x_3) = (\bar{x}_1 \bar{x}_2, x_3) = R(x_1 x_2 x_3), \end{aligned}$$

which together with $\alpha E = E$, shows that $\alpha \in F_4$ and $\alpha E_i = E_i, i = 1, 2, 3$. Therefore $\alpha \in D_4$. Certainly φ is a homomorphism. We shall show that φ is onto. Let $\alpha \in D_4$. We put

$$\mathfrak{J}_i = \{F_i(x) \mid x \in \mathfrak{C}\} = \{X \in \mathfrak{J} \mid 2E_{i+1} \circ X = 2E_{i+2} \circ X = X\}, \quad i = 1, 2, 3.$$

Since $\alpha X \in \mathfrak{J}_i, X \in \mathfrak{J}_i$, α induces \mathbf{R} -isomorphisms $\alpha : \mathfrak{J}_i \rightarrow \mathfrak{J}_i$ and $\alpha_i : \mathfrak{C} \rightarrow \mathfrak{C}$ satisfying

$$\alpha F_i(x) = F_i(\alpha_i x), \quad x \in \mathfrak{C},$$

for $i = 1, 2, 3$. Applying α on $F_i(x) \circ F_i(y) = (x, y)(E_{i+1} + E_{i+2})$, since the left side becomes $F_i(\alpha_i x) \circ F_i(\alpha_i y) = (\alpha_i x, \alpha_i y)(E_{i+1} + E_{i+2})$, we have

$$(\alpha_i x, \alpha_i y) = (x, y), \quad x, y \in \mathfrak{C}.$$

Hence $\alpha_i \in O(8), i = 1, 2, 3$. Moreover, by applying α on $F_1(x) \circ F_2(y) = \frac{1}{2}F_3(\bar{x}y)$, we see that

$$(\alpha_1 x)(\alpha_2 y) = \overline{\alpha_3(\bar{x}y)}, \quad x, y \in \mathfrak{C}.$$

From Lemma 1.14.4, we have $\alpha_1, \alpha_2, \alpha_3 \in SO(8)$, so that $(\alpha_1, \alpha_2, \alpha_3) \in Spin(8)$ and $\varphi(\alpha_1, \alpha_2, \alpha_3) = \alpha$. Therefore φ is onto. Evidently $\text{Ker } \varphi = \{(1, 1, 1)\}$. Consequently φ is an isomorphism.

We shall study the following subgroup $(F_4)_{E_1}$ of F_4 :

$$(F_4)_{E_1} = \{\alpha \in F_4 \mid \alpha E_1 = E_1\}.$$

We define \mathbf{R} -vector subspaces $\mathfrak{J}_{01}, \mathfrak{J}_{23}$ of \mathfrak{J} respectively by

$$\mathfrak{J}_{01} = \{X \in \mathfrak{J} \mid E_1 \circ X = 0, \text{tr}(X) = 0\} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & \xi & x \\ 0 & \bar{x} & -\xi \end{pmatrix} \mid \xi \in \mathbf{R}, x \in \mathfrak{C} \right\},$$

$$\mathfrak{J}_{23} = \{Y \in \mathfrak{J} \mid 2E_1 \circ Y = Y\} = \left\{ \begin{pmatrix} 0 & y_3 & \bar{y}_2 \\ \bar{y}_3 & 0 & 0 \\ y_2 & 0 & 0 \end{pmatrix} \mid y_2, y_3 \in \mathfrak{C} \right\}.$$

Lemma 2.7.2. *Suppose that we are given an element $A \in \mathfrak{J}_{01}$ such that $(A, A) = 2$. Choose any element $X_0 \in \mathfrak{J}_{01}$ such that $(X_0, X_0) = 2$, $(A, X_0) = 0$. Choose any element $Y_0 \in \mathfrak{J}_{23}$ such that $(Y_0, Y_0) = 2$, $2A \circ Y_0 = -Y_0$. Let*

$$Z_0 = 2X_0 \circ Y_0.$$

Choose any element $X_1 \in \mathfrak{J}_{01}$ such that $(X_1, X_1) = 2$, $(A, X_1) = (X_0, X_1) = 0$. Choose any element $X_2 \in \mathfrak{J}_{01}$ such that $(X_2, X_2) = 2$, $(A, X_2) = (X_0, X_2) = (X_1, X_2) = 0$. Let

$$Y_1 = -2Z_0 \circ X_1, \quad Z_2 = -2X_2 \circ Y_0, \quad X_3 = -2Y_1 \circ Z_2.$$

Finally, choose any element $X_4 \in \mathfrak{J}_{01}$ such that $(X_4, X_4) = 2$, $(A, X_4) = (X_0, X_4) = (X_1, X_4) = (X_2, X_4) = (X_3, X_4) = 0$. Let

$$\begin{aligned} Z_4 &= -2X_4 \circ Y_0, & Y_2 &= -2Z_0 \circ X_2, & Y_3 &= -2Z_0 \circ X_3, \\ X_5 &= -2Y_1 \circ Z_4, & X_6 &= 2Y_2 \circ Z_4, & X_7 &= -2Y_3 \circ Z_4 \end{aligned}$$

and moreover let

$$\begin{aligned} Y_i &= -2Z_0 \circ X_i, & i &= 4, 5, 6, 7, \\ Z_i &= -2X_i \circ Y_0, & i &= 1, 3, 5, 6, 7. \end{aligned}$$

Then, an \mathbf{R} -linear mapping $\alpha : \mathfrak{J} \rightarrow \mathfrak{J}$ satisfying

$$\begin{aligned} \alpha E &= E, & \alpha E_1 &= E_1, & \alpha(E_2 - E_3) &= A, \\ \alpha F_1(e_i) &= X_i, & \alpha F_2(e_i) &= Y_i, & \alpha F_3(e_i) &= Z_i, & i &= 0, 1, \dots, 7 \end{aligned}$$

belongs to $(F_4)_{E_1}$.

Proof. We have to show that the setting of the lemma is appropriate and prove that α satisfies

$$\alpha(X \circ Y) = \alpha X \circ \alpha Y, \quad X, Y \in \mathfrak{J}.$$

For this purpose, it is sufficient to show that this holds for generators $X, Y = E_i, F_j(e_k)$ of \mathfrak{J} . We have to check $27^2 = 729$ times if honestly doing so we reduce the number of times after preparing some lemma. However we omit its proof. In details, see Yokota [40].

Proposition 2.7.3. $(F_4)_{E_1} / \text{Spin}(8) \simeq S^8$.

In particular, the group $(F_4)_{E_1}$ is connected.

Proof. $S^8 = \{X \in \mathfrak{J}_{01} \mid (X, X) = 2\}$ is an 8 dimensional sphere. For $\alpha \in (F_4)_{E_1}$ and $X \in S^8$, we have $\alpha X \in S^8$. Hence the group $(F_4)_{E_1}$ acts on S^8 . This action is transitive. Indeed, for a given $A \in S^8$, by constructing $\alpha \in (F_4)_{E_1}$ of Lemma 2.7.2,

we have $\alpha(E_2 - E_3) = A$, and which shows the transitivity. We determine the isotropy subgroup of $(F_4)_{E_1}$ at $E_2 - E_3 \in S^8$. Let $\alpha \in (F_4)_{E_1}$ satisfy $\alpha(E_2 - E_3) = E_2 - E_3$. Since $\alpha \in (F_4)_{E_1}$ satisfies $\alpha E_1 = E_1$ and $\alpha E = E$, it also satisfies $\alpha(E_2 + E_3) = E_2 + E_3$. Therefore we have $\alpha E_2 = E_2$ and $\alpha E_3 = E_3$, so that $\alpha \in Spin(8)$. Conversely, $\alpha \in Spin(8)$ satisfies $\alpha(E_2 - E_3) = E_2 - E_3$. Thus we have the homeomorphism $(F_4)_{E_1}/Spin(8) \simeq S^8$.

Remark. If we know the dimension of the group $(F_4)_{E_1}$, without using Lemma 2.7.2, Proposition 2.7.3 can be simply proved as follows. The group $(F_4)_{E_1}$ acts on S^8 . The isotropy subgroup of $(F_4)_{E_1}$ at $E_2 - E_3$ is $Spin(8)$ and $\dim((F_4)_{E_1}/Spin(8)) = \dim(F_4)_{E_1} - \dim Spin(8) = 36 - 28 = \dim S^8$. Therefore, we have $(F_4)_{E_1}/Spin(8) \simeq S^8$.

Theorem 2.7.4. $(F_4)_{E_1} \cong Spin(9)$.

(From now on, we identify these groups).

Proof. Let $O(9) = O(\mathfrak{J}_{01}) = \{\alpha' \in \text{Iso}_{\mathbf{R}}(\mathfrak{J}_{01}) \mid (\alpha'X, \alpha'Y) = (X, Y)\}$. Consider the restriction $\alpha' = \alpha|_{\mathfrak{J}_{01}}$ of $\alpha \in (F_4)_{E_1}$ to \mathfrak{J}_{01} , then $\alpha' \in O(9)$. Hence we can define a homomorphism $p : (F_4)_{E_1} \rightarrow O(9)$ by $p(\alpha) = \alpha'$. Since p is continuous and $(F_4)_{E_1}$ is connected (Proposition 2.7.3), the mapping p induces a homomorphism

$$p : (F_4)_{E_1} \rightarrow SO(9).$$

We shall show p is onto. Let $SO(8) = \{\alpha' \in SO(9) \mid \alpha'(E_2 - E_3) = E_2 - E_3\}$. The restriction p' of $p : (F_4)_{E_1} \rightarrow SO(9)$ to $Spin(8) = \{\alpha \in (F_4)_{E_1} \mid \alpha(E_2 - E_3) = E_2 - E_3\}$ coincides with the homomorphism $p : Spin(9) \rightarrow SO(8)$ in Theorem 1.16.2. In particular, $p' : Spin(8) \rightarrow SO(8)$ is onto. Hence, from the following commutative diagram

$$\begin{array}{ccccccc} 1 & \longrightarrow & Spin(8) & \longrightarrow & (F_4)_{E_1} & \longrightarrow & S^8 \longrightarrow * \\ & & \downarrow p' & & \downarrow p & & \downarrow = \\ 1 & \longrightarrow & SO(8) & \longrightarrow & SO(9) & \longrightarrow & S^8 \longrightarrow * \end{array}$$

we see that $p : (F_4)_{E_1} \rightarrow SO(9)$ is onto by the five lemma. $\text{Ker } p = \{1, \sigma\}$, where $\sigma = \varphi(1, -1, -1)$ (which corresponds to σ defined in the following Section 2.9). Indeed, let $\alpha \in \text{Ker } p$, then α satisfies $\alpha X = X$ for all $X \in \mathfrak{J}_{01}$. From $\alpha(E_2 - E_3) = E_2 - E_3$ follows that $\alpha \in Spin(8)$. Let $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in Spin(8)$. Since $\alpha F_1(x) = F_1(x)$, that is, $F_1(\alpha_1 x) = F_1(x)$, so $\alpha_1 x = x$ for all $x \in \mathfrak{C}$, hence we have $\alpha_1 = 1$. From the principle of triality, we have $\alpha = (1, 1, 1) = 1$ or $\alpha = (1, -1, -1) = \sigma$. Hence we have $\text{Ker } p = \{1, \sigma\}$, and so we have the isomorphism

$$(F_4)_{E_1}/\{1, \sigma\} \cong SO(9).$$

Therefore $(F_4)_{E_1}$ is isomorphic to the group $Spin(9)$ as the universal covering group of $SO(9)$.

Theorem 2.7.5.

$$Spin(9)/Spin'(7) \simeq S^{15}.$$

where $Spin'(7) = \{\tilde{\alpha} \in SO(8) \mid (\tilde{\alpha}x)(\alpha y) = \tilde{\alpha}(xy), x, y \in \mathfrak{C} \text{ for some } \alpha \in SO(7)\}$.

Proof. Let $S^{15} = \{Y \in \mathfrak{J}_{23} \mid (Y, Y) = 2\}$. For $\alpha \in Spin(9)$ and $Y \in S^{15}$, we have $\alpha Y \in S^{15}$. Hence $Spin(9)$ acts on S^{15} . We shall show that this action is transitive. Let $Y_0 \in S^{15}$. Choose any element $A \in \mathfrak{J}_{01}$ such that $(A, A) = 2$, $2A \circ Y_0 = -Y_0$. Using these A and Y_0 , construct X_i, Y_i, Z_i and α of Lemma 2.7.2, then $\alpha \in Spin(9)$ and satisfies $\alpha F_2(1) = Y_0$, which shows the transitivity. We determine the isotropy subgroup $Spin(9)_{F_2(1)}$ of $Spin(9)$ at $F_2(1) \in S^{15}$. Let $\alpha \in Spin(9)$ satisfies $\alpha F_2(1) = F_2(1)$. Applying α on $F_2(1) \circ F_2(1) = E_1 + E_3$ we have $F_2(1) \circ F_2(1) = E_1 + \alpha E_3$, so we get $\alpha E_3 = E_3$. Since α always satisfies $\alpha(E_2 + E_3) = E_2 + E_3$, we have $\alpha E_2 = E_2$, and so $\alpha \in Spin(8)$. Denote $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, then $F_2(\alpha_2(1)) = F_2(1)$ implies $\alpha_2 1 = 1$, so that $\alpha_2 \in SO(7)$. Hence $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in Spin'(7)$. Conversely $\alpha \in Spin(7)$ satisfies $\alpha F_2(1) = F_2(1)$. Therefore $Spin(9)_{F_2(1)} = Spin'(7)$ is proved. Thus we have the homeomorphism $Spin(9)/Spin'(7) \simeq S^{15}$.

$(Spin'(7)$ and $Spin(7)$ are conjugate in the group $O(7)$. Indeed, a mapping $f : Spin'(7) \rightarrow Spin(7)$,

$$f(\alpha') = \varepsilon \alpha' \varepsilon^{-1}, \quad (\varepsilon x = \bar{x}, x \in \mathfrak{C})$$

gives the conjugation. In particular, $Spin'(7) \cong Spin(7)$

2.8. Connectedness of F_4

We denote by $(F_4)_0$ the connected component of F_4 containing the identity 1.

Lemma 2.8.1. For $a \in \mathfrak{C}$, we define a mapping $\beta_1(a) : \mathfrak{J} \rightarrow \mathfrak{J}$ by $\beta_1(a)X(\xi, x) = Y(\eta, y)$, where

$$\begin{cases} \eta_1 = \xi_1 \\ \eta_2 = \frac{\xi_2 + \xi_3}{2} + \frac{\xi_2 - \xi_3}{2} \cos 2|a| + \frac{(a, x_1)}{|a|} \sin 2|a| \\ \eta_3 = \frac{\xi_2 + \xi_3}{2} - \frac{\xi_2 - \xi_3}{2} \cos 2|a| - \frac{(a, x_1)}{|a|} \sin 2|a|, \\ \begin{cases} y_1 = x_1 - \frac{(\xi_2 - \xi_3)a}{2|a|} \sin 2|a| - \frac{2(a, x_1)a}{|a|^2} \sin^2 |a| \\ y_2 = x_2 \cos |a| - \frac{\bar{x}_3 \bar{a}}{|a|} \sin |a| \\ y_3 = x_3 \cos |a| + \frac{\bar{a} x_2}{|a|} \sin |a| \end{cases} \end{cases}$$

(if $a = 0$, then $\frac{\sin |a|}{|a|}$ means 1), then $\beta_1(a) \in (F_4)_0$.

Proof. For $A_1(a) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & a \\ 0 & -\bar{a} & 0 \end{pmatrix}$, we have $\tilde{A}_1(a) \in \mathfrak{f}_4$ (Proposition 2.3.6) and

$\beta_1(a) = \exp \tilde{A}_1(a)$. Hence $\beta_1(a) \in (F_4)_0$.

Proposition 2.8.2. *Any element $X \in \mathfrak{J}$ can be transformed to a diagonal form by some element $\alpha \in (F_4)_0$:*

$$\alpha X = \begin{pmatrix} \xi_1 & 0 & 0 \\ 0 & \xi_2 & 0 \\ 0 & 0 & \xi_3 \end{pmatrix}, \quad \xi_i \in \mathbf{R}.$$

Moreover, ξ_1, ξ_2, ξ_3 are uniquely determined (up to their permutation) independent of the choice of $\alpha \in (F_4)_0$.

Proof. For a given $X \in \mathfrak{J}$, consider a space $\mathfrak{X} = \{\alpha X \mid \alpha \in (F_4)_0\}$. Since $(F_4)_0$ is compact (Theorem 2.2.5), \mathfrak{X} is also compact. Let $\xi_1^2 + \xi_2^2 + \xi_3^2$ be the maximal value of all $\eta_1^2 + \eta_2^2 + \eta_3^2$ for $Y = Y(\eta, y) \in \mathfrak{X}$ and let $X_0 = X(\xi, x)$ be an element of \mathfrak{X} which attains its maximal value. Then X_0 is of diagonal form. Certainly, suppose X_0 is not of diagonal form, for example, the 2×3 entry x_1 of X_0 is non-zero: $x_1 \neq 0$. Let $a(t) = \frac{x_1}{|x_1|}t$, $t > 0$, and construct $\beta_1(a(t)) \in (F_4)_0$ of Lemma 2.8.1. Since $|a(t)| = t$ and $\frac{(a(t), x_1)}{|a(t)|} = |x_1|$ for $Y(\eta(t), y(t)) = \beta_1(a(t))X_0 \in \mathfrak{X}$, we have

$$\begin{aligned} & \eta_1(t)^2 + \eta_2(t)^2 + \eta_3(t)^2 \\ &= \xi_1^2 + \left(\frac{\xi_2 + \xi_3}{2} + \frac{\xi_2 - \xi_3}{2} \cos 2t + |x_1| \sin 2t \right)^2 \\ & \quad + \left(\frac{\xi_2 + \xi_3}{2} - \frac{\xi_2 - \xi_3}{2} \cos 2t - |x_1| \sin 2t \right)^2 \\ &= \xi_1^2 + 2 \left(\frac{\xi_2 + \xi_3}{2} \right)^2 + 2 \left(\frac{\xi_2 - \xi_3}{2} \cos 2t + |x_1| \sin 2t \right)^2 \\ &= \xi_1^2 + 2 \left(\frac{\xi_2 + \xi_3}{2} \right)^2 + 2 \left(\left(\frac{\xi_2 - \xi_3}{2} \right)^2 + |x_1|^2 \right) \sin^2(2t + t_0) \text{ (for some } t_0 \in \mathbf{R}) \\ &\leq \xi_1^2 + 2 \left(\frac{\xi_2 + \xi_3}{2} \right)^2 + 2 \left(\left(\frac{\xi_2 - \xi_3}{2} \right)^2 + |x_1|^2 \right) \\ &= \xi_1^2 + \xi_2^2 + \xi_3^2 + 2|x_1|^2 \end{aligned}$$

which is the maximal value and attains at some $t > 0$. This contradicts the maximum of $\xi_1^2 + \xi_2^2 + \xi_3^2$. Hence $x_1 = 0$. $x_2 = x_3 = 0$ can be similarly proved by constructing $\beta_2(a), \beta_3(a) \in (F_4)_0$ analogous to $\beta_1(a)$ of Lemma 2.8.1. Hence X_0 is of diagonal form. We now give the proof of the latter half of the proposition. If $X \in \mathfrak{J}$ is transformed

to a diagonal form $\alpha X = \begin{pmatrix} \xi_1 & 0 & 0 \\ 0 & \xi_2 & 0 \\ 0 & 0 & \xi_3 \end{pmatrix}$ by $\alpha \in (F_4)_0$, then

$$\begin{aligned} \text{tr}(X) &= \text{tr}(\alpha X) \text{ (Lemma 2.2.1.(2))} = \sum_{i=1}^3 \xi_i, \\ (X, X) &= (\alpha X, \alpha X) \text{ (Lemma 2.2.4)} = \sum_{i=1}^3 \xi_i^2, \\ \text{tr}(X, X, X) &= \text{tr}(\alpha X, \alpha X, \alpha X) \text{ (Lemma 2.2.4)} = \sum_{i=1}^3 \xi_i^3. \end{aligned}$$

Hence, ξ_1, ξ_2, ξ_3 are uniquely determined (up to order) as the solutions of the following simultaneous equation:

$$\begin{cases} \xi_1 + \xi_2 + \xi_3 = \text{tr}(X) \\ \xi_1^2 + \xi_2^2 + \xi_3^2 = (X, X) \\ \xi_1^3 + \xi_2^3 + \xi_3^3 = \text{tr}(X, X, X). \end{cases}$$

The space \mathfrak{CP}_2 , called the Cayley projective plane, is defined as

$$\mathfrak{CP}_2 = \{X \in \mathfrak{J} \mid X^2 = X, \text{tr}(X) = 1\}.$$

Theorem 2.8.3. $F_4/\text{Spin}(9) \simeq \mathfrak{CP}_2$.

In particular, the group F_4 is connected.

Proof. For $\alpha \in F_4$ and $X \in \mathfrak{CP}_2$, we have $\alpha X \in \mathfrak{CP}_2$. Hence F_4 acts on \mathfrak{CP}_2 . We shall prove that the group $(F_4)_0$ acts transitively on \mathfrak{CP}_2 . To prove this, it is sufficient to show that any element $X \in \mathfrak{CP}_2$ can be transformed to $E_1 \in \mathfrak{CP}_2$ by some $\alpha \in (F_4)_0$. Now, $X \in \mathfrak{CP}_2 \subset \mathfrak{J}$ can be transformed to a diagonal form by $\alpha \in (F_4)_0$:

$$\alpha X = \begin{pmatrix} \xi_1 & 0 & 0 \\ 0 & \xi_2 & 0 \\ 0 & 0 & \xi_3 \end{pmatrix}, \quad \xi_i \in \mathbf{R}$$

(Proposition 2.8.2). From the condition $X \circ X = X$, we have $\alpha X \circ \alpha X = \alpha X$, that is,

$$\begin{pmatrix} \xi_1^2 & 0 & 0 \\ 0 & \xi_2^2 & 0 \\ 0 & 0 & \xi_3^2 \end{pmatrix} = \begin{pmatrix} \xi_1 & 0 & 0 \\ 0 & \xi_2 & 0 \\ 0 & 0 & \xi_3 \end{pmatrix}.$$

Hence $\xi_i^2 = \xi_i$, so that $\xi_i = 1$ or $\xi_i = 0$, $i = 1, 2, 3$. Next, from $\text{tr}(\alpha X) = \text{tr}(X) = 1$, we have $\xi_i = 1$, $\xi_{i+1} = \xi_{i+2} = 0$ for some i , that is, $\alpha X = E_i$. Moreover, E_2, E_3 are transformed to E_1 respectively by $(F_4)_0$. Certainly, if we define a mapping

$\beta : \mathfrak{J} \rightarrow \mathfrak{J}$, $\beta X = T X T^{-1}$, where $T = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \in SO(3)$, then $\beta \in (F_4)_0$ and

$\beta E_2 = E_1$. Hence $\beta \alpha X = E_1$. In the case $\alpha X = E_3$, the situation is similar to the above. Therefore the transitivity is proved. Since we have $\mathfrak{CP}_2 = (F_4)_0 E_1$, \mathfrak{CP}_2 is connected. Now, the group F_4 acts transitively on \mathfrak{CP}_2 and the isotropy subgroup of F_4 at $E_1 \in \mathfrak{CP}_2$ is $\text{Spin}(9)$ (Theorem 2.7.4). Thus we have the homeomorphism $F_4/\text{Spin}(9) \simeq \mathfrak{CP}_2$. Finally, the connectedness of F_4 follows from the connectedness of \mathfrak{CP}_2 and $\text{Spin}(9)$.

2.9. Involution σ and subgroup $\text{Spin}(9)$ of F_4

Definition. We define an \mathbf{R} -linear transformation σ of \mathfrak{J} by

$$\sigma \begin{pmatrix} \xi_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & \xi_2 & x_1 \\ x_2 & \bar{x}_1 & \xi_3 \end{pmatrix} = \begin{pmatrix} \xi_1 & -x_3 & -\bar{x}_2 \\ -\bar{x}_3 & \xi_2 & x_1 \\ -x_2 & \bar{x}_1 & \xi_3 \end{pmatrix}.$$

Then $\sigma \in F_4$ and $\sigma^2 = 1$. Observe that this σ is the same as $\sigma = (1, -1, -1) \in Spin(8) \subset Spin(9) \subset F_4$ of Theorem 2.7.4.

We shall study the following subgroup $(F_4)^\sigma$ of F_4 :

$$(F_4)^\sigma = \{\alpha \in F_4 \mid \sigma\alpha = \alpha\sigma\}.$$

For this end, we consider \mathbf{R} -vector subspaces \mathfrak{J}_σ and $\mathfrak{J}_{-\sigma}$ of \mathfrak{J} , which are eigenspaces of σ , respectively by

$$\begin{aligned} \mathfrak{J}_\sigma &= \{X \in \mathfrak{J} \mid \sigma X = X\} = \left\{ \begin{pmatrix} \xi_1 & 0 & 0 \\ 0 & \xi_2 & x_1 \\ 0 & \bar{x}_1 & \xi_3 \end{pmatrix} \mid \xi_i \in \mathbf{R}, x_1 \in \mathfrak{C} \right\} \\ &= \{X \in \mathfrak{J} \mid E_1 \circ X = 0\} \oplus \mathfrak{C}_1 \quad (\text{where } \mathfrak{C}_1 = \{\xi E_1 \mid \xi \in \mathbf{R}\}) \\ &= \mathfrak{J}(2, \mathfrak{C}) \oplus \mathfrak{C}_1, \\ \mathfrak{J}_{-\sigma} &= \{X \in \mathfrak{J} \mid \sigma X = -X\} = \left\{ \begin{pmatrix} 0 & x_3 & \bar{x}_2 \\ \bar{x}_3 & 0 & 0 \\ x_2 & 0 & 0 \end{pmatrix} \mid x_i \in \mathfrak{C} \right\} \\ &= \{X \in \mathfrak{J} \mid 2E_1 \circ X = X\}. \end{aligned}$$

Then, $\mathfrak{J} = \mathfrak{J}_\sigma \oplus \mathfrak{J}_{-\sigma}$ and $\mathfrak{J}_\sigma, \mathfrak{J}_{-\sigma}$ are invariant under the action of $(F_4)^\sigma$.

Theorem 2.9.1. $(F_4)^\sigma = (F_4)_{E_1} = Spin(9)$.

Proof. We shall show that for $\alpha \in (F_4)^\sigma$ we have $\alpha E_1 = E_1$. Let $\alpha \in (F_4)^\sigma$. We first show that

$$\alpha E_2, \alpha E_3 \in \mathfrak{J}(2, \mathfrak{C}).$$

Certainly we have

$$\begin{aligned} \alpha E_2 &= \alpha(-F_2(1) \times F_2(1)) = -\alpha F_2(1) \times \alpha F_2(1) \\ &= -(F_2(x_2) + F_3(x_3)) \times (F_2(x_2) + F_3(x_3)) \quad (\text{for some } x_2, x_3 \in \mathfrak{C}) \\ &= (x_2, x_2)E_2 + (x_3, x_3)E_3 - F_1(\overline{x_2 x_3}) \in \mathfrak{J}(2, \mathfrak{C}). \end{aligned}$$

Similarly $\alpha E_3 \in \mathfrak{J}(2, \mathfrak{C})$. So $\alpha E_1 = \alpha(E - E_2 - E_3) = E - \alpha E_2 - \alpha E_3$ is of the form

$$\alpha E_1 = E_1 + \xi_2 E_2 + \xi_3 E_3 + F_1(x_1).$$

Then we have

$$1 = (E_1, E_1) = (\alpha E_1, \alpha E_1) = 1 + \xi_2^2 + \xi_3^2 + 2|x_1|^2,$$

so that $\xi_2 = \xi_3 = x_1 = 0$, therefore $\alpha E_1 = E_1$, that is, $\alpha \in (F_4)_{E_1}$. Conversely let $\alpha \in (F_4)_{E_1}$, that is, $\alpha \in F_4$ satisfies $\alpha E_1 = E_1$. Noting that $\mathfrak{J} = \mathfrak{J}_\sigma \oplus \mathfrak{J}_{-\sigma}$ and $\mathfrak{J}_\sigma = \{X \in \mathfrak{J} \mid E_1 \circ X = 0\} \oplus \mathfrak{E}_1$, $\mathfrak{J}_{-\sigma} = \{X \in \mathfrak{J} \mid 2E_1 \circ X = X\}$ are invariant under α , we have

$$\begin{aligned} \alpha\sigma X &= \alpha\sigma(X_1 + X_2) & X_1 \in \mathfrak{J}_\sigma, \quad X_2 \in \mathfrak{J}_{-\sigma} \\ &= \alpha(X_1 - X_2) = \alpha X_1 - \alpha X_2 = \sigma(\alpha X_1) + \sigma(\alpha X_2) \\ &= \sigma\alpha(X_1 + X_2) = \sigma\alpha X, \quad X \in \mathfrak{J}. \end{aligned}$$

Thus $\alpha\sigma = \sigma\alpha$, and so $\alpha \in (F_4)^\sigma$. Therefore we have shown that $(F_4)^\sigma = (F_4)_{E_1} \cong Spin(9)$ (Theorem 2.7.4).

2.10. Center $z(F_4)$ of F_4

Theorem 2.10.1. *The center $z(F_4)$ of the group F_4 is trivial:*

$$z(F_4) = \{1\}.$$

Proof. Let $\alpha \in z(F_4)$. From the commutativity with σ : $\sigma\alpha = \alpha\sigma$, we have $\alpha \in Spin(9)$ (Theorem 2.9.1) and so $\alpha \in z(Spin(9))$. Since the center $z(Spin(9))$ of $Spin(9)$ is the group of order 2, we have

$$\alpha = 1 \quad \text{or} \quad \alpha = \sigma.$$

However, $\sigma \notin z(F_4)$ (Theorem 2.9.1), which implies that $\alpha = 1$.

According to a general theory of compact Lie groups, it is known that the center of the simply connected simple Lie group of type F_4 is trivial. Hence F_4 has to be simply connected. Thus we have the following Theorem.

Theorem 2.10.2. $F_4 = \{\alpha \in \text{Iso}_{\mathbf{R}}(\mathfrak{J}) \mid \alpha(X \circ Y) = \alpha X \circ \alpha Y\}$ is a simply connected compact Lie group of type F_4 .

Remark. If we know that the space \mathfrak{CP}_2 is simply connected, the simply connectedness of the group F_4 follows from $F_4/Spin(9) \simeq \mathfrak{CP}_2$ of Theorem 2.8.3.

2.11. Involution γ and subgroup $(Sp(1) \times Sp(3))/Z_2$ of F_4

We define an involutive \mathbf{R} -linear transformation γ of \mathfrak{J} by

$$\gamma \begin{pmatrix} \xi_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & \xi_2 & x_1 \\ x_2 & \bar{x}_1 & \xi_3 \end{pmatrix} = \begin{pmatrix} \xi_1 & \gamma x_3 & \overline{\gamma x_2} \\ \overline{\gamma x_3} & \xi_2 & \gamma x_1 \\ \gamma x_2 & \overline{\gamma x_1} & \xi_3 \end{pmatrix}.$$

This γ is the same as $\gamma \in G_2 \subset F_4$.

We shall study the following subgroup $(F_4)^\gamma$ of F_4 :

$$(F_4)^\gamma = \{\alpha \in F_4 \mid \gamma\alpha = \alpha\gamma\}.$$

Any element $X \in \mathfrak{J}$ is expressed by

$$X = \begin{pmatrix} \xi_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & \xi_2 & x_1 \\ x_2 & \bar{x}_1 & \xi_3 \end{pmatrix} = \begin{pmatrix} \xi_1 & m_3 & \bar{m}_2 \\ \bar{m}_3 & \xi_2 & m_1 \\ m_2 & \bar{m}_1 & \xi_3 \end{pmatrix} + \begin{pmatrix} 0 & a_3 e_4 & -a_2 e_4 \\ -a_3 e_4 & 0 & a_1 e_4 \\ a_2 e_4 & -a_1 e_4 & 0 \end{pmatrix},$$

where $x_i = m_i + a_i e_4 \in \mathbf{H} \oplus \mathbf{H}e_4 = \mathfrak{C}$. We associate such element $X \in \mathfrak{J}$ with the element of $\mathfrak{J}(3, \mathbf{H}) \oplus \mathbf{H}^3$,

$$\begin{pmatrix} \xi_1 & m_3 & \bar{m}_2 \\ \bar{m}_3 & \xi_2 & m_1 \\ m_2 & \bar{m}_1 & \xi_3 \end{pmatrix} + (a_1, a_2, a_3).$$

In $\mathfrak{J}(3, \mathbf{H}) \oplus \mathbf{H}^3$, we define a multiplication \times , an inner product $(\ , \)$ and an \mathbf{R} -linear transformation γ respectively by

$$\begin{aligned} (M + \mathbf{a}) \times (N + \mathbf{b}) &= \left(M \times N - \frac{1}{2}(\mathbf{a}^* \mathbf{b} + \mathbf{b}^* \mathbf{a}) \right) - \frac{1}{2}(\mathbf{a}N + \mathbf{b}M), \\ (M + \mathbf{a}, N + \mathbf{b}) &= (M, N) + 2(\mathbf{a}, \mathbf{b}), \\ \gamma(M + \mathbf{a}) &= M - \mathbf{a}. \end{aligned}$$

(In $\mathfrak{J}(3, \mathbf{H})$ the multiplication $M \times N$ and the inner product (M, N) are analogously defined as in \mathfrak{J} , and in \mathbf{H}^3 the inner product (\mathbf{a}, \mathbf{b}) is defined naturally by $\frac{1}{2}(\mathbf{a}^* \mathbf{b} + \mathbf{b}^* \mathbf{a})$). Since these operations correspond to their respective operations in \mathfrak{J} , hereafter, we identify $\mathfrak{J}(3, \mathbf{H}) \oplus \mathbf{H}^3$ with \mathfrak{J} , that is,

$$\mathfrak{J}(3, \mathbf{H}) \oplus \mathbf{H}^3 = \mathfrak{J}.$$

The group $F_{4, \mathbf{H}}$ is defined to be the automorphism group of the Jordan algebra $\mathfrak{J}_{\mathbf{H}} = \mathfrak{J}(3, \mathbf{H})$ (in which multiplications \circ and \times are analogously defined as in \mathfrak{J}):

$$\begin{aligned} F_{4, \mathbf{H}} &= \{\alpha \in \text{Iso}_{\mathbf{R}}(\mathfrak{J}_{\mathbf{H}}) \mid \alpha(M \circ N) = \alpha M \circ \alpha N\} \\ &= \{\alpha \in \text{Iso}_{\mathbf{R}}(\mathfrak{J}_{\mathbf{H}}) \mid \alpha(M \times N) = \alpha M \times \alpha N\}. \end{aligned}$$

Proposition 2.11.1. $F_{4, \mathbf{H}} \cong Sp(3)/\mathbf{Z}_2$, $\mathbf{Z}_2 = \{E, -E\}$.

Proof. We define a mapping $\varphi : Sp(3) \rightarrow F_{4, \mathbf{H}}$ by

$$\varphi(A)M = AMA^*, \quad M \in \mathfrak{J}_{\mathbf{H}}.$$

It is not difficult to see that φ is well-defined and is a homomorphism. We shall show that φ is onto. For a given $\alpha \in F_{4, \mathbf{H}}$, we consider αE_i , $i = 1, 2, 3$. Since αE_i satisfies $\alpha E_i \circ \alpha E_i = \alpha E_i$ and $\text{tr}(\alpha E_i) = 1$ (that is, $\alpha E_i \in \mathbf{HP}_2 = \{M \in \mathfrak{J}_{\mathbf{H}} \mid M \circ M =$

$M, \text{tr}(M) = 1\}$ (the quaternion projective plane)), there exists $A_i \in Sp(3)$ such that $\alpha E_i = A_i E_i A_i^*$. Let $\mathbf{a}_i = \begin{pmatrix} a_{i1} \\ a_{i2} \\ a_{i3} \end{pmatrix}$ be the i -th column vector of A_i , then we have $\alpha E_i = \begin{pmatrix} a_{i1}\bar{a}_{i1} & a_{i1}\bar{a}_{i2} & a_{i1}\bar{a}_{i3} \\ a_{i2}\bar{a}_{i1} & a_{i2}\bar{a}_{i2} & a_{i2}\bar{a}_{i3} \\ a_{i3}\bar{a}_{i1} & a_{i3}\bar{a}_{i2} & a_{i3}\bar{a}_{i3} \end{pmatrix}$. Construct a matrix $A = (\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3)$, then we have

$$\alpha E_i = A E_i A^*, \quad i = 1, 2, 3.$$

Since $AA^* = A(E_1 + E_2 + E_3)A^* = \alpha E_1 + \alpha E_2 + \alpha E_3 = \alpha E = E$, we have $A \in Sp(3)$. If we let $\beta = \varphi(A)^{-1}\alpha$, then $\beta \in F_{4,\mathbf{H}}$ and satisfies

$$\beta E_i = E_i, \quad i = 1, 2, 3.$$

Analogously as in Theorem 2.7.1, β induces orthogonal transformations $\beta_1, \beta_2, \beta_3 : \mathbf{H} \rightarrow \mathbf{H}$ satisfying

$$(\beta_1 m)(\beta_2 n) = \overline{\beta_3(\overline{mn})}, \quad m, n \in \mathbf{H}. \quad (\text{i})$$

Put $p = \beta_1 1$ and $q = \beta_2 1$, then we have $|p| = |q| = 1$. Letting $m = 1$ and $n = 1$ in (i), we have $p(\beta_2 n) = \overline{\beta_3 \overline{n}}$ and $(\beta_1 m)q = \overline{\beta_3 \overline{m}}$, and hence

$$\beta_2 m = \overline{p}(\beta_1 m)q, \quad \beta_3 m = \overline{(\beta_1 \overline{m})}q.$$

Substitute $\zeta m = \overline{p}(\beta_1 m)$ in (i), then we see that ζ satisfies

$$(\zeta m)(\zeta n) = \zeta(mn), \quad m, n \in \mathbf{H},$$

that is, ζ is an automorphism of \mathbf{H} : $\zeta \in \text{Aut}(\mathbf{H})$. Hence, ζ is expressed by $\zeta m = rm\bar{r}$, using $r \in Sp(1)$ (Proposition 108 of Yokota [58]). Consequently

$$\beta_1 m = prm\bar{r}, \quad \beta_2 m = rm\bar{r}q, \quad \beta_3 m = \bar{q}rm\bar{r}\bar{p}, \quad m \in \mathbf{H}.$$

Construct a matrix $B = \begin{pmatrix} \bar{q}r & 0 & 0 \\ 0 & pr & 0 \\ 0 & 0 & r \end{pmatrix}$, then $B \in Sp(3)$ and we have

$$\beta M = BMB^*, \quad M \in \mathfrak{J}_{\mathbf{H}},$$

that is, $\beta = \varphi(B)$. Hence

$$\alpha = \varphi(A)\beta = \varphi(A)\varphi(B) = \varphi(AB), \quad AB \in Sp(3).$$

Therefore φ is onto. $\text{Ker } \varphi = \{E, -E\}$ can be easily obtained. Thus we have the isomorphism $Sp(3)/\mathbf{Z}_2 \cong F_{4,\mathbf{H}}$.

Theorem 2.11.2. $(F_4)^\gamma \cong (Sp(1) \times Sp(3))/\mathbf{Z}_2$, $\mathbf{Z}_2 = \{(1, E), (-1, -E)\}$.

Proof. We define a mapping $\varphi : Sp(1) \times Sp(3) \rightarrow (F_4)^\gamma$ by

$$\varphi(p, A)(M + \mathbf{a}) = AMA^* + p\mathbf{a}A^*, \quad M + \mathbf{a} \in \mathfrak{J}_H \oplus \mathbf{H}^3 = \mathfrak{J}.$$

We first show that $\varphi(p, A) \in (F_4)^\gamma$. For $p \in Sp(1)$, $A \in Sp(3)$, $M, N \in \mathfrak{J}_H$ and $\mathbf{a}, \mathbf{b} \in \mathbf{H}^3$, we have

$$\begin{aligned} AMA^* \times ANA^* &= A(M \times N)A^*, \\ (p\mathbf{a}A^*)^*(p\mathbf{b}A^*) &= A\mathbf{a}^*\bar{p}p\mathbf{b}A^* = A(\mathbf{a}^*\mathbf{b})A^*, \\ (p\mathbf{a}A^*)(ANA^*) &= p(\mathbf{a}N)A^*, \text{ etc.} \end{aligned}$$

From which, we see that $\varphi(p, A)$ satisfies

$$\varphi(p, A)((M + \mathbf{a}) \times (N + \mathbf{b})) = \varphi(p, A)(M + \mathbf{a}) \times \varphi(p, A)(N + \mathbf{b}),$$

hence $\varphi(p, A) \in F_4$. Clearly $\gamma\varphi(p, A) = \varphi(p, A)\gamma$, so that $\varphi(p, A) \in (F_4)^\gamma$. Certainly φ is a homomorphism. We shall show that φ is onto. Let $\alpha \in (F_4)^\gamma$. Since the restriction $\alpha' = \alpha|_{\mathfrak{J}_H}$ of α to $\mathfrak{J}_H = \{X \in \mathfrak{J} \mid \gamma X = X\}$ belongs to $F_{4, H}$, there exists $A \in Sp(3)$ such that

$$\alpha M = AMA^*, \quad M \in \mathfrak{J}_H$$

(Proposition 2.11.1). Let $\beta = \varphi(1, A)^{-1}\alpha$, then $\beta|_{\mathfrak{J}_H} = 1$. Therefore $\beta \in G_2$. Certainly, since $\beta \in Spin(7)$, by letting $\beta = (\beta_1, \beta, \kappa\beta)$, we see that $(\beta_1 x)(\beta y) = \beta(xy)$. Now let $y = 1$, then we see $\beta_1 = \beta$. Since $\beta \in (G_2)^\gamma$ and $\beta|_H = 1$, there exists $p \in Sp(1)$ such that

$$\beta(m + ae_4) = m + (pa)e_4, \quad m + ae_4 \in \mathbf{H} \oplus \mathbf{H}e_4 = \mathfrak{C},$$

(Theorem 1.10.1). Hence we have

$$\beta(M + \mathbf{a}) = M + p\mathbf{a} = \varphi(p, E)(M + \mathbf{a}),$$

so that $\beta = \varphi(p, E)$. Hence we have $\alpha = \varphi(1, A)\beta = \varphi(1, A)\varphi(p, E) = \varphi(p, A)$. Therefore φ is onto. $\text{Ker } \varphi = \{(1, E), (-1, -E)\} = \mathbf{Z}_2$ can be easily obtained. Thus we have the isomorphism $(Sp(1) \times Sp(3))/\mathbf{Z}_2 \cong (F_4)^\gamma$.

Remark. Since $(F_4)^\gamma$ is connected as fixed points subgroup of F_4 by the involution γ of the simply connected Lie group F_4 , the fact that $\varphi : Sp(1) \times Sp(3) \rightarrow F_4$ is onto can be proved as follows. The elements

$$\begin{aligned} G_{ij}, \quad 0 \leq i < j \leq 3, 4 \leq i < j \leq 7, \\ \tilde{A}_k(e_j), \quad 0 \leq j \leq 3, k = 1, 2, 3 \end{aligned}$$

forms an \mathbf{R} -basis of $(\mathfrak{f}_4)^\gamma$. So $\dim((\mathfrak{f}_4)^\gamma) = 6 \times 2 + 4 \times 3 = 24 = 3 + 21 = \dim(\mathfrak{sp}(1) \oplus \mathfrak{sp}(3))$. Hence φ is onto.

2.12. Automorphism w of order 3 and subgroup $(SU(3) \times SU(3))/\mathbf{Z}_3$ of F_4

We define an \mathbf{R} -linear transformation w of \mathfrak{J} by

$$w \begin{pmatrix} \xi_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & \xi_2 & x_1 \\ x_2 & \bar{x}_1 & \xi_3 \end{pmatrix} = \begin{pmatrix} \xi_1 & \omega x_3 & \overline{\omega x_2} \\ \overline{\omega x_3} & \xi_2 & \omega x_1 \\ \omega x_2 & \overline{\omega x_1} & \xi_3 \end{pmatrix}.$$

This w is the same as $w \in G_2 \subset F_4$.

We shall study the following subgroup $(F_4)^w$ of F_4 :

$$(F_4)^w = \{\alpha \in F_4 \mid w\alpha = \alpha w\}.$$

We associate an element

$$\begin{pmatrix} \xi_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & \xi_2 & x_1 \\ x_2 & \bar{x}_1 & \xi_3 \end{pmatrix}, \quad \xi_i \in \mathbf{R}, x_i = a_i + \mathbf{m}_i \in \mathbf{C} \oplus \mathbf{C}^3 = \mathfrak{C}$$

of \mathfrak{J} with the element

$$\begin{pmatrix} \xi_1 & a_3 & \bar{a}_2 \\ \bar{a}_3 & \xi_2 & a_1 \\ a_2 & \bar{a}_1 & \xi_3 \end{pmatrix} + (\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3)$$

of $\mathfrak{J}(3, \mathbf{C}) \oplus M(3, \mathbf{C})$. In $M(3, \mathbf{C})$, we define the exterior product $M \times N$ by

$$M \times N = \begin{pmatrix} \mathbf{m}_2 \times \mathbf{n}_3 & \mathbf{m}_3 \times \mathbf{n}_1 & \mathbf{m}_1 \times \mathbf{n}_2 \\ + & + & + \\ \mathbf{n}_2 \times \mathbf{m}_3 & \mathbf{n}_3 \times \mathbf{m}_1 & \mathbf{n}_1 \times \mathbf{m}_2 \end{pmatrix} \in M(3, \mathbf{C}),$$

where $M = (\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3), N = (\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3) \in M(3, \mathbf{C})$. Then, for $P, A \in M(3, \mathbf{C})$ and $M, N \in M(3, \mathbf{C})$, we have

$$PM \times PN = {}^t\tilde{P}(M \times N), \quad MA \times NA = (M \times N) {}^t\tilde{A},$$

where \tilde{P} and \tilde{A} are the adjoint matrices of P and A , respectively. Further, in $M(3, \mathbf{C})$, we define a real valued symmetric product (M, N) by

$$(M, N) = \frac{1}{2} \text{tr}(M^*N + N^*M) = \sum_{i,j} (m_{ij}, n_{ij}),$$

where $M = (m_{ij}), N = (n_{ij}) \in M(3, \mathbf{C})$.

In $\mathfrak{J}(3, \mathbf{C}) \oplus M(3, \mathbf{C})$, we define the multiplication \times , the inner product $(\ , \)$ and the \mathbf{R} -linear transformation w respectively by

$$\begin{aligned} (X + M) \times (Y + N) &= (X \times Y - \frac{1}{2}(M^*N + N^*M)) - \frac{1}{2}(MY + NX + \overline{M \times N}), \\ (X + M, Y + N) &= (X, Y) + 2(M, N), \\ w(X + M) &= X + \omega_1 M, \quad \left(\omega_1 = -\frac{1}{2} + \frac{\sqrt{3}}{2}e_1\right), \end{aligned}$$

(in $\mathfrak{J}(3, \mathbf{C})$, multiplications \circ and \times are defined as in \mathfrak{J}). Since these operations correspond to their respective operations in \mathfrak{J} , we identify $\mathfrak{J}(3, \mathbf{C}) \oplus M(3, \mathbf{C})$ with \mathfrak{J} , that is,

$$\mathfrak{J}(3, \mathbf{C}) \oplus M(3, \mathbf{C}) = \mathfrak{J}.$$

The group $F_{4, \mathbf{C}}$ is defined to be the automorphism group of the Jordan algebra $\mathfrak{J}_{\mathbf{C}} = \mathfrak{J}(3, \mathbf{C})$:

$$\begin{aligned} F_{4, \mathbf{C}} &= \{\alpha \in \text{Iso}_{\mathbf{R}}(\mathfrak{J}_{\mathbf{C}}) \mid \alpha(X \circ Y) = \alpha X \circ \alpha Y\} \\ &= \{\alpha \in \text{Iso}_{\mathbf{R}}(\mathfrak{J}_{\mathbf{C}}) \mid \alpha(X \times Y) = \alpha X \times \alpha Y\}. \end{aligned}$$

The group $\mathbf{Z}_2 = \{1, \epsilon\}$ acts on the group $SU(3)$ by

$$\epsilon A = \overline{A}, \quad A \in SU(3),$$

and the group $SU(3) \cdot \mathbf{Z}_2$ be the semi-direct product of groups $SU(3)$ and \mathbf{Z}_2 under this action.

Proposition 2.12.1. $F_{4, \mathbf{C}} \cong (SU(3)/\mathbf{Z}_3) \cdot \mathbf{Z}_2$, $\mathbf{Z}_3 = \{E, \omega_1 E, \omega_1^2 E\}$, $\omega_1 = -\frac{1}{2} + \frac{\sqrt{3}}{2}e_1$.

Proof. We define a mapping $\varphi : SU(3) \cdot \mathbf{Z}_2 \rightarrow F_{4, \mathbf{C}}$ by

$$\varphi(A, 1)X = AXA^*, \quad \varphi(A, \epsilon)X = A\overline{X}A^*, \quad X \in \mathfrak{J}_{\mathbf{C}}.$$

It is not difficult to see that φ is well-defined and that it is a homomorphism. The proof that φ is onto is the same as that of Proposition 2.11.1 (use $C, SU(3)$, instead of $\mathbf{H}, Sp(3)$), and we only need to modify a little the last part. Since $\zeta \in \text{Aut}(\mathbf{C})$ can be either $\zeta = 1$ or $\zeta = \epsilon$ (where $\epsilon x = \overline{x}$, $x \in \mathbf{C}$), we have

$$\beta_1 x = px, \beta_2 x = xq, \beta_3 x = \overline{q}x\overline{p} \quad \text{or} \quad \beta_1 x = p\overline{x}, \beta_2 x = \overline{x}q, \beta_3 x = \overline{q}\overline{x}\overline{p}.$$

Now, choose $r \in \mathbf{C}$ such that $r^{-3} = \overline{q}p$ and construct a matrix $B = \begin{pmatrix} \overline{q}r & 0 & 0 \\ 0 & pr & 0 \\ 0 & 0 & r \end{pmatrix}$.

Then, $B \in SU(3)$ and we have

$$\beta X = BXB^* \quad \text{or} \quad \beta X = B\overline{X}B^*, \quad X \in \mathfrak{J}_{\mathbf{C}}.$$

The remaining proof is again analogous to that of Proposition 2.11.1.

Theorem 2.12.2. $(F_4)^w \cong (SU(3) \times SU(3))/\mathbf{Z}_3$, $\mathbf{Z}_3 = \{(E, E), (\omega_1 E, \omega_1 E), (\omega_1^2 E, \omega_1^2 E)\}$, $\omega_1 = -\frac{1}{2} + \frac{\sqrt{3}}{2}e_1$.

Proof. We define a mapping $\varphi : SU(3) \times SU(3) \rightarrow (F_4)^w$ by

$$\varphi(P, A)(X + M) = AXA^* + PMA^*, \quad X + M \in \mathfrak{J}(3, \mathbf{C}) \oplus M(3, \mathbf{C}) = \mathfrak{J}.$$

We first show that $\varphi(P, A) \in (F_4)^w$. For $P, A \in SU(3)$ and $X + M, Y + N \in \mathfrak{J}(3, \mathbf{C}) \oplus M(3, \mathbf{C})$, we have

$$\begin{aligned} AXA^* \times AYA^* &= A(X \times Y)A^*, \\ (PMA^*)^*(PNA^*) &= AM^*P^*PNA = A(M^*N)A, \\ (PMA^*)(AYA^*) &= P(MY)A^*, \\ \overline{PMA^* \times PNA^*} &= \overline{{}^t\tilde{P}(M \times N){}^t\tilde{A}^*} = \overline{PM \times NA^*}, \text{ etc.} \end{aligned}$$

Using them, we see that $\varphi(P, A)$ satisfies

$$\varphi(P, A)((X + M) \times (Y + N)) = \varphi(P, A)(X + M) \times \varphi(P, A)(Y + N),$$

hence $\varphi(P, A) \in F_4$. Clearly $w\varphi(P, A) = \varphi(P, A)w$, so that $\varphi(P, A) \in (F_4)^w$. Certainly φ is a homomorphism. We shall show that φ is onto. Let $\alpha \in (F_4)^w$. Since the restriction $\alpha' = \alpha|_{\mathfrak{J}_C}$ of α to $\mathfrak{J}_C = \{X \in \mathfrak{J} \mid wX = X\}$ belongs to $F_{4,C}$, there exists $A \in SU(3)$ such that

$$\alpha X = AXA^* \quad \text{or} \quad \alpha X = A\overline{X}A^*, \quad X \in \mathfrak{J}_C$$

(Proposition 2.12.1). In the former case, let $\beta = \varphi(E, A)^{-1}\alpha$, then $\beta|_{\mathfrak{J}_C} = 1$, and so $\beta \in G_2$. Moreover $\beta \in (G_2)_{e_1} = (G_2)^w = SU(3)$ (Theorem 1.9.4), and hence, there exists $P \in SU(3)$ such that

$$\beta(X + M) = X + PM = \varphi(P, E)(X + M), \quad X + M \in \mathfrak{J}_C \oplus M(3, \mathbf{C}) = \mathfrak{J}.$$

Therefore we have

$$\alpha = \varphi(E, A)\beta = \varphi(E, A)\varphi(P, E) = \varphi(P, A).$$

In the latter case, consider the mapping $\gamma_1 : \mathfrak{J} \rightarrow \mathfrak{J}$ given by $\gamma_1(X + M) = \overline{X} + \overline{M}$, $X + M \in \mathfrak{J}$ and remember that $\gamma_1 \in G_2 \subset F_4$. Let $\beta = \alpha^{-1}\varphi(E, A)\gamma_1$. Then $\beta \in F_4$ and $\beta|_{\mathfrak{J}_C} = 1$, which shows that $\beta \in (G_2)_{e_1} = (G_2)^w$ (Theorem 1.9.4) $\subset (F_4)^w$. Since $\alpha, \varphi(E, A) \in (F_4)^w$, we have $\gamma_1 \in (F_4)^w$, so that $\gamma_1 \in (G_2)^w$ which is a contradiction (Theorem 1.9.4). Consequently the proof of φ is onto is completed. The fact that $\text{Ker}\varphi = \{(E, E), (\omega_1 E, \omega_1 E), (\omega_1^2 E, \omega_1^2 E)\} = \mathbf{Z}_3$ can be easily obtained. Thus we have the isomorphism $(SU(3) \times SU(3))/\mathbf{Z}_3 \cong (F_4)^w$.

Remark 1. Since $(F_4)^w$ is connected as the fixed points subgroup of F_4 by an automorphism of order 3 of the simply connected group F_4 (Rasevskii [32]), that $\varphi : SU(3) \times SU(3) \rightarrow (F_4)^w$ is onto is proved as follows. The elements

$$\begin{aligned} G_{01}, \quad G_{23}, \quad G_{45}, \quad G_{67}, \quad G_{26} + G_{37}, \quad -G_{27} + G_{36}, \\ G_{24} + G_{35}, \quad -G_{25} + G_{34}, \quad G_{46} + G_{57}, \quad -G_{47} + G_{56}, \\ \tilde{A}_1(1), \quad \tilde{A}_2(1), \quad \tilde{A}_3(1), \quad \tilde{A}_1(e_1), \quad \tilde{A}_2(e_1), \quad \tilde{A}_3(e_1) \end{aligned}$$

forms an \mathbf{R} -basis of $(\mathfrak{f}_4)^w$. So $\dim(\mathfrak{f}_4)^w = 10 + 6 = 16 = 8 + 8 = \dim(\mathfrak{su}(3) + \mathfrak{su}(3))$. Hence φ is onto.

Remark 2. The group F_4 has a subgroup which is isomorphic to the group $(SU(3) \times SU(3))/\mathbf{Z}_3 \cdot \mathbf{Z}_2$, which is the semi-direct product of the groups $(SU(3) \times SU(3))/\mathbf{Z}_3$ and \mathbf{Z}_2 (the action of $\mathbf{Z}_2 = \{1, \gamma_1\}$ on the group $(SU(3) \times SU(3))$ is $\gamma_1(P, A) = (\overline{P}, \overline{A})$).

2.13. Complex exceptional Lie group F_4^C

Definition. The group F_4^C is defined to be the automorphism group of the complex Jordan algebra \mathfrak{J}^C :

$$\begin{aligned} F_4^C &= \{\alpha \in \text{Iso}_C(\mathfrak{J}^C) \mid \alpha(X \circ Y) = \alpha X \circ \alpha Y\} \\ &= \{\alpha \in \text{Iso}_C(\mathfrak{J}^C) \mid \det(\alpha X) = \det X, (\alpha X, \alpha Y) = (X, Y)\} \\ &= \{\alpha \in \text{Iso}_C(\mathfrak{J}^C) \mid \det(\alpha X) = \det X, \alpha E = E\} \\ &= \{\alpha \in \text{Iso}_C(\mathfrak{J}^C) \mid \alpha(X \times Y) = \alpha X \times \alpha Y\}. \end{aligned}$$

We define a positive definite Hermitian inner product $\langle X, Y \rangle$ in \mathfrak{J}^C by

$$\langle X, Y \rangle = (\tau X, Y).$$

For $\alpha \in \text{Hom}_C(\mathfrak{J}^C)$, we denote the complex conjugate transpose of α with respect to $\langle X, Y \rangle$ by α^* : $\langle \alpha^* X, Y \rangle = \langle X, \alpha Y \rangle$.

Lemma 2.13.1. (1) For $\alpha \in F_4^C$, we have $\alpha^* = \tau \alpha^{-1} \tau \in F_4^C$.

(2) For any $\alpha \in F_4$, its complexified mapping $\alpha^C : \mathfrak{J}^C \rightarrow \mathfrak{J}^C$ belongs to F_4^C : $\alpha^C \in F_4^C$. Identifying α with α^C , we regard F_4 as a subgroup of F_4^C : $F_4 \subset F_4^C$. For $\alpha \in F_4^C$, we have $\alpha \in F_4$ if and only if $\tau \alpha = \alpha \tau$, that is,

$$F_4 = \{\alpha \in F_4^C \mid \tau \alpha = \alpha \tau\}.$$

Proof. (1) $\langle \alpha^* X, Y \rangle = \langle X, \alpha Y \rangle = (\tau X, \alpha Y) = (\alpha^{-1} \tau X, Y) = \langle \tau \alpha^{-1} \tau X, Y \rangle$ for all $X, Y \in \mathfrak{J}^C$. Hence $\alpha^* = \tau \alpha^{-1} \tau \in F_4^C$.

(2) Let $\alpha \in F_4^C$ satisfy $\tau \alpha = \alpha \tau$. Then, since $\tau \alpha X = \alpha \tau X = \alpha X$, we have $\alpha X \in \mathfrak{J}$ for $X \in \mathfrak{J}$. Hence α induces an \mathbf{R} -transformation α' of \mathfrak{J} and $\alpha' \in F_4$, further we have $\alpha = (\alpha')^C$.

Theorem 2.13.2. The polar decomposition of the Lie group F_4^C is given by

$$F_4^C \simeq F_4 \times \mathbf{R}^{52}.$$

In particular, F_4^C is a simply connected complex Lie group of type F_4 .

Proof. Evidently F_4^C is an algebraic subgroup of $\text{Iso}_C(\mathfrak{J}^C) = GL(27, C)$. If $\alpha \in F_4^C$, then $\alpha^* \in F_4^C$ (Lemma 2.13.1.(1)). Hence from Chevalley's lemma, we have

$$F_4^C \simeq (F_4^C \cap U(\mathfrak{J}^C)) \times \mathbf{R}^d = F_4 \times \mathbf{R}^d,$$

where $U(\mathfrak{J}^C) = \{\alpha \in \text{Iso}_C(\mathfrak{J}^C) \mid \langle \alpha X, \alpha Y \rangle = \langle X, Y \rangle\}$ and $d = \dim F_4^C - \dim F_4 = 2 \times 52 - 52 = 52$. Since F_4 is simply connected (Theorem 2.10.2), F_4^C is also simply connected. The Lie algebra of the group F_4^C is \mathfrak{f}_4^C , so that F_4^C is a complex simple Lie group of type F_4 .

2.14. Non-compact exceptional Lie groups $F_{4(4)}$ and $F_{4(-20)}$ of type F_4

Consider the following two \mathbf{R} -vector spaces

$$\begin{aligned}\mathfrak{J}(3, \mathfrak{C}') &= \{X \in M(3, \mathfrak{C}') \mid X^* = X\}, \\ \mathfrak{J}(1, 2, \mathfrak{C}) &= \{X \in M(3, \mathfrak{C}) \mid I_1 X^* I_1 = X\},\end{aligned}$$

where $I_1 = \text{diag}(-1, 1, 1) \in M(3, \mathbf{R})$. We define the Jordan multiplication $X \circ Y$ in $\mathfrak{J}(3, \mathfrak{C}')$ and $\mathfrak{J}(1, 2, \mathfrak{C})$ respectively by

$$X \circ Y = \frac{1}{2}(XY + YX).$$

Then we have the isomorphisms

$$\begin{aligned}\mathfrak{J}(3, \mathfrak{C}') &\cong \{X \in \mathfrak{J}(3, \mathfrak{C}') \mid \tau\gamma X = X\} = (\mathfrak{J}(3, \mathfrak{C}'))_{\tau\gamma}, \\ \mathfrak{J}(1, 2, \mathfrak{C}) &\cong \{X \in \mathfrak{J}(3, \mathfrak{C}') \mid \tau\sigma X = X\} = (\mathfrak{J}(3, \mathfrak{C}'))_{\tau\sigma}\end{aligned}$$

as Jordan algebras, therefore we identify them, respectively. We denote by $F_{4(4)}$ and $F_{4(-20)}$ respectively the automorphism groups of the Jordan algebras $\mathfrak{J}(3, \mathfrak{C}')$ and $\mathfrak{J}(1, 2, \mathfrak{C})$:

$$\begin{aligned}F_{4(4)} &= \{\alpha \in \text{Iso}_{\mathbf{R}}(\mathfrak{J}(3, \mathfrak{C}')) \mid \alpha(X \circ Y) = \alpha X \circ \alpha Y\}, \\ F_{4(-20)} &= \{\alpha \in \text{Iso}_{\mathbf{R}}(\mathfrak{J}(1, 2, \mathfrak{C})) \mid \alpha(X \circ Y) = \alpha X \circ \alpha Y\}.\end{aligned}$$

They can also be defined by

$$F_{4(4)} = (F_4^C)^{\tau\gamma}, \quad F_{4(-20)} = (F_4^C)^{\tau\sigma}.$$

Theorem 2.14.1. *The polar decompositions of the Lie groups $F_{4(4)}$ and $F_{4(-20)}$ are respectively given by*

$$\begin{aligned}F_{4(4)} &\simeq (Sp(1) \times Sp(3))/\mathbf{Z}_2 \times \mathbf{R}^{28}, \\ F_{4(-20)} &\simeq Spin(9) \times \mathbf{R}^{16}.\end{aligned}$$

Proof. These are facts corresponding to Theorems 2.11.2 and 2.9.1.

Theorem 2.14.2. *The centers $z(F_{4(4)})$ and $z(F_{4(-20)})$ are trivial:*

$$z(F_{4(4)}) = \{1\}, \quad z(F_{4(-20)}) = \{1\}.$$

Exceptional Lie group E_6

3.1. Compact exceptional Lie group E_6

Let \mathfrak{J}^C be the complex exceptional Jordan algebra.

Definition. We define the groups E_6^C and E_6 respectively by

$$\begin{aligned} E_6^C &= \{\alpha \in \text{Iso}_C(\mathfrak{J}^C) \mid \det(\alpha X) = \det X\} \\ &= \{\alpha \in \text{Iso}_C(\mathfrak{J}^C) \mid (\alpha X, \alpha Y, \alpha Z) = (X, Y, Z)\}, \\ E_6 &= \{\alpha \in \text{Iso}_C(\mathfrak{J}^C) \mid \det(\alpha X) = \det X, \langle \alpha X, \alpha Y \rangle = \langle X, Y \rangle\} \\ &= \{\alpha \in \text{Iso}_C(\mathfrak{J}^C) \mid (\alpha X, \alpha Y, \alpha Z) = (X, Y, Z), \langle \alpha X, \alpha Y \rangle = \langle X, Y \rangle\} \\ &= \{\alpha \in \text{Iso}_C(\mathfrak{J}^C) \mid \alpha X \times \alpha Y = {}^t\alpha^{-1}(X \times Y), \langle \alpha X, \alpha Y \rangle = \langle X, Y \rangle\} \\ &= \{\alpha \in \text{Iso}_C(\mathfrak{J}^C) \mid \alpha X \times \alpha Y = \tau \alpha \tau(X \times Y), \langle \alpha X, \alpha Y \rangle = \langle X, Y \rangle\}. \end{aligned}$$

If we define an involutive automorphism λ of the group E_6^C by

$$\lambda(\alpha) = {}^t\alpha^{-1}, \quad \alpha \in E_6^C$$

(Lemma 2.2.3), then the definition of the group E_6 can be also given by

$$E_6 = \{\alpha \in E_6^C \mid \tau \lambda(\alpha) \tau = \alpha\} = (E_6^C)^{\tau \lambda}.$$

Theorem 3.1.1. E_6 is a compact Lie group.

Proof. E_6 is a compact Lie group as a closed subgroup of the unitary group

$$U(27) = U(\mathfrak{J}^C) = \{\alpha \in \text{Iso}_C(\mathfrak{J}^C) \mid \langle \alpha X, \alpha Y \rangle = \langle X, Y \rangle\}.$$

3.2. Lie algebra \mathfrak{e}_6 of E_6

Before investigating the Lie algebra \mathfrak{e}_6 of the group E_6 , we will study the Lie algebra \mathfrak{e}_6^C of the group E_6^C .

Theorem 3.2.1. (1) The Lie algebra \mathfrak{e}_6^C of the Lie group E_6^C is given by

$$\mathfrak{e}_6^C = \{\phi \in \text{Hom}_C(\mathfrak{J}^C) \mid (\phi X, X, X) = 0\}.$$

(2) Any element $\phi \in \mathfrak{e}_6^C$ is uniquely expressed by

$$\phi = \delta + \tilde{T}, \quad \delta \in \mathfrak{f}_4^C, T \in \mathfrak{J}_0^C.$$

In particular, the dimension of \mathfrak{e}_6^C is

$$\dim_C(\mathfrak{e}_6^C) = 52 + 26 = 78.$$

Proof. (1) is proved as similar way to Lemma 2.3.1.

(2) For $\phi \in \mathfrak{e}_6^C$, by letting $T = \phi E$, we obtain $T \in \mathfrak{J}^C$, and $\text{tr}(T) = 0$. Certainly $\text{tr}(T) = (T, E, E) = (\phi E, E, E) = 0$. If we put $\delta = \phi - \tilde{T}$, then $\delta \in \mathfrak{e}_6^C$. Moreover $\delta \in \mathfrak{f}_4^C$, because $\delta E = \phi E - \tilde{T}E = T - T = 0$. Hence we have $\phi = \delta + \tilde{T}$. To prove the uniqueness of the expression, it is sufficient to show that

$$\delta + \tilde{T} = 0, \quad \delta \in \mathfrak{f}_4^C, T \in \mathfrak{J}_0^C \quad \text{implies} \quad \delta = 0, T = 0.$$

Certainly, let apply it on E , then we have $T = 0$, so that $\delta = 0$. Finally, we have $\dim_C(\mathfrak{e}_6^C) = 52 + 26 = 78$ from the expression above. Therefore the theorem is proved.

Theorem 3.2.2. *The Lie bracket $[\phi_1, \phi_2]$ in \mathfrak{e}_6^C is given by*

$$[\delta_1 + \tilde{T}_1, \delta_2 + \tilde{T}_2] = ([\delta_1, \delta_2] + [\tilde{T}_1, \tilde{T}_2]) + (\widetilde{\delta_1 T_2} - \widetilde{\delta_2 T_1}),$$

where $\phi_i = \delta_i + \tilde{T}_i, \delta_i \in \mathfrak{f}_4^C, T_i \in \mathfrak{J}_0^C$.

Proof. It is sufficient to show that $[\delta, \tilde{T}] = \widetilde{\delta T}$ for $\delta \in \mathfrak{f}_4^C, T \in \mathfrak{J}_0^C$. Now,

$$\begin{aligned} [\delta, \tilde{T}]X &= \delta(T \circ X) - T \circ \delta X = \delta T \circ X + T \circ \delta X - T \circ \delta X \\ &= \delta T \circ X = \widetilde{\delta T}X, \quad X \in \mathfrak{J}^C. \end{aligned}$$

We shall investigate the Lie algebra \mathfrak{e}_6 of the Lie group E_6 .

Lemma 3.2.3. *For $\phi = \delta + \tilde{T} \in \mathfrak{e}_6^C, \delta \in \mathfrak{f}_4^C, T \in \mathfrak{J}_0^C$, we have*

$$\lambda(\phi) = -^t\phi = -^t(\delta + \tilde{T}) = \delta - \tilde{T}.$$

In particular, $-^t\phi \in \mathfrak{e}_6^C$.

Proof. $(-^t\phi X, Y) = -(X, \phi Y) = -(X, \delta Y + \tilde{T}Y) = -(X, \delta Y) - (X, T \circ Y)$
 $= (\delta X, Y) - (\tilde{T}X, Y) = ((\delta - \tilde{T})X, Y) \quad X, Y \in \mathfrak{J}^C.$

Therefore $-^t\phi = \delta - \tilde{T} \in \mathfrak{e}_6^C$.

Theorem 3.2.4. (1) *The Lie algebra \mathfrak{e}_6 of the Lie group E_6 is given by*

$$\mathfrak{e}_6 = \{\phi \in \text{Hom}_C(\mathfrak{J}^C) \mid (\phi X, X, X) = 0, \langle \phi X, Y \rangle + \langle X, \phi Y \rangle = 0\}.$$

(2) *Any element $\phi \in \mathfrak{e}_6$ is uniquely expressed by*

$$\phi = \delta + i\tilde{T}, \quad \delta \in \mathfrak{f}_4, T \in \mathfrak{J}_0.$$

Proof. (1) The proof is evident (cf. Lemma 2.3.1).

(2) For $\phi \in \mathfrak{e}_6^C$, the condition $\phi \in \mathfrak{e}_6$ is equivalent to $\tau\lambda(\phi)\tau = \phi$. Now, if ϕ is of the form $\phi = \delta + \tilde{T}', \delta \in \mathfrak{f}_4^C, T' \in \mathfrak{J}_0^C$ (Theorem 3.2.1.(2)), then $\tau\lambda(\phi)\tau = \phi$ is

$\tau\delta\tau - \widetilde{\tau T'} = \delta + \widetilde{T'}$ (Lemma 3.2.3), that is, $\tau\delta\tau = \delta$ and $\tau T' = -T'$. Hence $\delta \in \mathfrak{f}_4$, and T' is of the form $T' = iT, T \in \mathfrak{J}_0$.

Proposition 3.2.5. *The complexification of the Lie algebra \mathfrak{e}_6 is \mathfrak{e}_6^C .*

Proof. For $\phi \in \mathfrak{e}_6^C$, the conjugate transposed mapping ϕ^* of ϕ with respect to the inner product $\langle X, Y \rangle$ of \mathfrak{J}^C is $\phi^* = \tau^t \phi \tau \in \mathfrak{e}_6^C$, and for $\phi \in \mathfrak{e}_6^C$, ϕ belongs to \mathfrak{e}_6 if and only if $\phi^* = -\phi$. Now, any element $\phi \in \mathfrak{e}_6^C$ can be uniquely expressed as

$$\phi = \frac{\phi - \phi^*}{2} + i \frac{\phi + \phi^*}{2i}, \quad \frac{\phi - \phi^*}{2}, \frac{\phi + \phi^*}{2i} \in \mathfrak{e}_6.$$

Hence \mathfrak{e}_6^C is the complexification of \mathfrak{e}_6 .

3.3. Simplicity of \mathfrak{e}_6^C

Theorem 3.3.1. *The Lie algebra \mathfrak{e}_6^C is simple and so \mathfrak{e}_6 is also simple.*

Proof. We use the decomposition of \mathfrak{e}_6^C of Theorem 3.2.1.(2):

$$\mathfrak{e}_6^C = \mathfrak{f}_4^C \oplus \widetilde{\mathfrak{J}}_0^C.$$

Let $p : \mathfrak{e}_6^C \rightarrow \mathfrak{f}_4^C$ and $q : \mathfrak{e}_6^C \rightarrow \widetilde{\mathfrak{J}}_0^C$ be projections of $\mathfrak{e}_6^C = \mathfrak{f}_4^C \oplus \widetilde{\mathfrak{J}}_0^C$. Now, let \mathfrak{a} be a non-zero ideal of \mathfrak{e}_6^C . Then $p(\mathfrak{a})$ is an ideal of \mathfrak{f}_4^C . Indeed, if $\delta \in p(\mathfrak{a})$, then there exists $T \in \widetilde{\mathfrak{J}}_0^C$ such that $\delta + \widetilde{T} \in \mathfrak{a}$. For any $\delta_1 \in \mathfrak{f}_4^C$, we have

$$\mathfrak{a} \ni [\delta_1, \delta + \widetilde{T}] = [\delta_1, \delta] + \widetilde{\delta_1 T} \quad (\text{Theorem 3.2.2}),$$

hence $[\delta_1, \delta] \in \mathfrak{a}$.

We shall show that either $\mathfrak{f}_4^C \cap \mathfrak{a} \neq \{0\}$ or $\widetilde{\mathfrak{J}}_0^C \cap \mathfrak{a} \neq \{0\}$. Assume that $\mathfrak{f}_4^C \cap \mathfrak{a} = \{0\}$ and $\widetilde{\mathfrak{J}}_0^C \cap \mathfrak{a} = \{0\}$. Then the mapping $p|_{\mathfrak{a}} : \mathfrak{a} \rightarrow \mathfrak{f}_4^C$ is injective because $\widetilde{\mathfrak{J}}_0^C \cap \mathfrak{a} = \{0\}$. Since $p(\mathfrak{a})$ is a non-zero ideal of \mathfrak{f}_4^C and \mathfrak{f}_4^C is simple, we have $p(\mathfrak{a}) = \mathfrak{f}_4^C$. Hence $\dim_C \mathfrak{a} = \dim_C p(\mathfrak{a}) = \dim_C \mathfrak{f}_4^C = 52$. On the other hand, since $\mathfrak{f}_4^C \cap \mathfrak{a} = \{0\}$, $q|_{\mathfrak{a}} : \mathfrak{a} \rightarrow \widetilde{\mathfrak{J}}_0^C$ is also injective, we have $\dim_C \mathfrak{a} \leq \dim_C \widetilde{\mathfrak{J}}_0^C = \dim_C \mathfrak{J}_0^C = 26$. This leads to a contradiction.

We now consider the following two cases.

(1) Case $\mathfrak{f}_4^C \cap \mathfrak{a} \neq \{0\}$. From the simplicity of \mathfrak{f}_4^C , we have $\mathfrak{f}_4^C \cap \mathfrak{a} = \mathfrak{f}_4^C$, hence $\mathfrak{a} \supset \mathfrak{f}_4^C$. On the other hand, we have

$$\mathfrak{a} \supset [\mathfrak{a}, \mathfrak{e}_6^C] \supset [\mathfrak{f}_4^C, \widetilde{\mathfrak{J}}_0^C] = \widetilde{\mathfrak{f}_4^C \mathfrak{J}_0^C} \quad (\text{Lemma 3.2.2}) = \widetilde{\mathfrak{J}}_0^C \quad (\text{Proposition 2.4.6.(2)}).$$

Consequently $\mathfrak{a} \supset \mathfrak{f}_4^C \oplus \widetilde{\mathfrak{J}}_0^C = \mathfrak{e}_6^C$.

(2) Case $\widetilde{\mathfrak{J}}_0^C \cap \mathfrak{a} \neq \{0\}$. Let \widetilde{A} ($A \in \mathfrak{J}_0^C$) be a non-zero element of $\widetilde{\mathfrak{J}}_0^C \cap \mathfrak{a} \subset \mathfrak{a}$. Choose $B \in \mathfrak{J}_0^C$ such that $[\widetilde{A}, \widetilde{B}] \neq 0$ (Lemma 2.5.4), then $0 \neq [\widetilde{A}, \widetilde{B}] \in \mathfrak{f}_4^C \cap \mathfrak{a}$. Hence this case is reduced to the case (1).

Therefore we have $\mathfrak{a} = \mathfrak{e}_6^C$.

Proposition 3.3.2. (1) \mathfrak{J}^C is a simple Jordan algebra.

(2) \mathfrak{J}^C is an \mathfrak{e}_6^C -irreducible C -module.

(3) $\mathfrak{e}_6^C \mathfrak{J}^C = \left\{ \sum_i \phi_i A_i \mid \phi_i \in \mathfrak{e}_6^C, A_i \in \mathfrak{J}^C \right\} = \mathfrak{J}^C$.

Proof. (1) Let \mathfrak{a} be a non-zero ideal of \mathfrak{J}^C and $X = X(\xi, x)$ a non-zero element of \mathfrak{a} .

(i) Case $\xi_1 \neq 0$. From $\xi_1 E_1 = (2X \circ E_1 - X) \circ E_1 \in \mathfrak{a}$, we have $E_1 \in \mathfrak{a}$. Next, from $F_2(1) = 2E_1 \circ F_2(1) \in \mathfrak{a}$ and $E_1 + E_3 = F_2(1) \circ F_2(1) \in \mathfrak{a}$, we have $E_3 = (E_1 + E_3) - E_1 \in \mathfrak{a}$. Similarly $E_2 \in \mathfrak{a}$, so that $E = E_1 + E_2 + E_3 \in \mathfrak{a}$. Now, for any $X \in \mathfrak{J}^C$, $X = E \circ X \in \mathfrak{a}$, and so $\mathfrak{a} = \mathfrak{J}^C$. In the case $\xi_2 \neq 0$ or $\xi_3 \neq 0$, the statement is also valid.

(ii) Case $\xi_1 = \xi_2 = \xi_3 = 0, x_1 \neq 0$. We have $F_1(x_1) = 4(X \circ E_2) \circ E_3 \in \mathfrak{a}$. Choose $a \in \mathfrak{C}^C$ such that $(x_1, a) = 1$, then $F_1(x_1) \circ F_1(a) = (x_1, a)(E_2 + E_3) = E_2 + E_3 \in \mathfrak{a}$. Hence this case is reduced to the case (i) and so $\mathfrak{a} = \mathfrak{J}^C$.

(2) Let W be a non-zero \mathfrak{e}_6^C -invariant C -submodule of \mathfrak{J}^C . For any $A \in \mathfrak{J}^C$ and $X \in W$, we have

$$A \circ X = \left(A - \frac{1}{3} \text{tr}(A) E \right)^\sim X + \frac{1}{3} \text{tr}(A) X \in W.$$

Hence, W is an ideal of \mathfrak{J}^C . From the simplicity of \mathfrak{J}^C ((1) above), we have $W = \mathfrak{J}^C$.

(3) $\mathfrak{e}_6^C \mathfrak{J}^C$ is an \mathfrak{e}_6^C -invariant C -submodule of \mathfrak{J}^C . Hence from the irreducibility of \mathfrak{J}^C ((2) above), we have $\mathfrak{e}_6^C \mathfrak{J}^C = \mathfrak{J}^C$.

3.4. Element $A \vee B$ of \mathfrak{e}_6^C

Definition. For $A, B \in \mathfrak{J}^C$, we define an element $A \vee B \in \mathfrak{e}_6^C$ by

$$A \vee B = [\tilde{A}, \tilde{B}] + \left(A \circ B - \frac{1}{3} (A, B) E \right)^\sim$$

(Proposition 2.4.1, Theorem 3.2.1).

Lemma 3.4.1. For $A, B \in \mathfrak{J}^C$, we have

$$(A \vee B)X = \frac{1}{2}(B, X)A + \frac{1}{6}(A, B)X - 2B \times (A \times X), \quad X \in \mathfrak{J}^C.$$

Proof. Consider Hamilton-Cayley formula $X \circ (X \times X) = (\det X)E$, $X \in \mathfrak{J}^C$, that is,

$$X \circ (X \circ X) - \text{tr}(X)X \circ X + \frac{1}{2}(\text{tr}(X)^2 - (X, X))X = \frac{1}{3}(X, X, X)E.$$

If we put $\lambda A + \mu B + \nu X$ in place of X , then taking the coefficient of $\lambda\mu\nu$, we have

$$\begin{aligned}
& A \circ (B \circ X) + B \circ (X \circ A) + X \circ (A \circ B) - \text{tr}(A)B \circ X - \text{tr}(B)A \circ X \\
& - \text{tr}(X)A \circ B + \frac{1}{2}(\text{tr}(A)\text{tr}(B)X + \text{tr}(B)\text{tr}(X)A + \text{tr}(X)\text{tr}(A)B) \\
& - \frac{1}{2}((A, B)X + (B, X)A + (X, A)B) = (A, B, X)E.
\end{aligned}$$

Therefore, using the above, we have

$$\begin{aligned}
& \frac{1}{2}(B, X)A + \frac{1}{6}(A, B)X - 2B \times (A \times X) + \frac{1}{3}(A, B)X \\
& = \frac{1}{2}(B, X)A + \frac{1}{2}(A, B)X - 2B \circ (A \times X) + \text{tr}(B)A \times X + \text{tr}(A \times X)B \\
& \quad - (\text{tr}(B)\text{tr}(A \times X) - (B, A \times X))E \\
& = \frac{1}{2}(B, X)A + \frac{1}{2}(A, B)X - 2B \circ (A \circ X) + \text{tr}(A)B \circ X + \text{tr}(X)B \circ A \\
& \quad - \text{tr}(A)\text{tr}(X)B + (A, X)B + \text{tr}(B)A \circ X - \frac{1}{2}\text{tr}(B)\text{tr}(A)X \\
& \quad - \frac{1}{2}\text{tr}(B)\text{tr}(X)A + \frac{1}{2}\text{tr}(B)\text{tr}(A)\text{tr}(X)E - \frac{1}{2}\text{tr}(B)(A, X)E \\
& \quad + \frac{1}{2}(\text{tr}(A)\text{tr}(X) - (A, X))B - \frac{1}{2}\text{tr}(B)(\text{tr}(A)\text{tr}(X) - (A, X))E + (A, B, X)E \\
& = \text{tr}(A)B \circ X + \text{tr}(B)X \circ A + \text{tr}(X)A \circ B \\
& \quad - \frac{1}{2}(\text{tr}(A)\text{tr}(B)X + \text{tr}(B)\text{tr}(X)A + \text{tr}(A)\text{tr}(X)B) \\
& \quad + \frac{1}{2}((A, B)X + (B, X)A + (A, X)B) + (A, B, X)E - 2B \circ (A \circ X) \\
& = A \circ (B \circ X) + B \circ (A \circ X) + X \circ (A \circ B) - 2B \circ (A \circ X) \\
& = A \circ (B \circ X) - B \circ (A \circ X) + (A \circ B) \circ X \\
& = [\tilde{A}, \tilde{B}]X + (A \circ B) \sim X.
\end{aligned}$$

Lemma 3.4.2. (1) $E_i \vee E_j = 0$, $i \neq j$.

(2) $E_1 \vee E_1 = \frac{1}{3}(2E_1 - E_2 - E_3)^\sim$, more explicitly, we have

$$(E_1 \vee E_1) \begin{pmatrix} \xi_1 & x_3 & \overline{x}_2 \\ \overline{x}_3 & \xi_2 & x_1 \\ x_2 & \overline{x}_1 & \xi_3 \end{pmatrix} = \frac{1}{6} \begin{pmatrix} 4\xi_1 & x_3 & \overline{x}_2 \\ \overline{x}_3 & -2\xi_2 & -2x_1 \\ x_2 & -2\overline{x}_1 & -2\xi_3 \end{pmatrix}.$$

For $\phi \in \text{Hom}_C(\mathfrak{J}^C)$, we often denote $-{}^t\phi$ by ϕ' :

$$(\phi'X, Y) = -(X, \phi Y), \quad X, Y \in \mathfrak{J}^C.$$

Lemma 3.4.3. (1) For $\phi \in \mathfrak{e}_6^C$, we have

$$\phi(X \times Y) = \phi'X \times Y + X \times \phi'Y, \quad X, Y \in \mathfrak{J}^C.$$

(2) For $A, B \in \mathfrak{J}^C$, we have

$$(A \vee B)' = -B \vee A.$$

Proof. (1) it is evident from Lemma 2.3.1.

$$(2) (A \vee B)' = \left([\tilde{A}, \tilde{B}] + \left(A \circ B - \frac{1}{3}(A, B)E \right)^\sim \right)' = [\tilde{A}, \tilde{B}] - \left(A \circ B - \frac{1}{3}(A, B)E \right)^\sim$$

$$(\text{Lemma 3.2.3}) = -[\tilde{B}, \tilde{A}] - \left(A \circ B - \frac{1}{3}(A, B)E \right)^\sim = -B \vee A.$$

Lemma 3.4.4. (1) For $\phi \in \mathfrak{e}_6^C$ and $A, B \in \mathfrak{J}^C$, we have

$$[\phi, A \vee B] = \phi A \vee B + A \vee \phi' B.$$

(2) Any element $\phi \in \mathfrak{e}_6^C$ is expressed by $\phi = \sum_i (A_i \vee B_i)$, $A_i, B_i \in \mathfrak{J}^C$.

Proof. (1) $[\phi, A \vee B]X = \phi(A \vee B)X - (A \vee B)\phi X$

$$= \phi \left(\frac{1}{2}(B, X)A + \frac{1}{6}(A, B)X - 2B \times (A \times X) \right) - (A \vee B)\phi X \quad (\text{Lemma 3.4.1})$$

$$= \frac{1}{2}(B, X)\phi A + \frac{1}{6}(A, B)\phi X - 2\phi' B \times (A \times X) - 2B \times (\phi A \times X)$$

$$- 2B \times (A \times \phi X) - \frac{1}{2}(B, \phi X)A - \frac{1}{6}(A, B)\phi X + 2B \times (A \times \phi X) \quad (\text{Lemma 3.4.3})$$

$$= \frac{1}{2}(B, X)\phi A + \frac{1}{6}(\phi A, B)X - 2B \times (\phi A \times X)$$

$$+ \frac{1}{2}(\phi' B, X)A + \frac{1}{6}(A, \phi' B)X - 2\phi' B \times (A \times X)$$

$$= (\phi A \vee B)X + (A \vee \phi' B)X, \quad X \in \mathfrak{J}^C.$$

(2) We can see from (1) that $\mathfrak{a} = \left\{ \sum_i (A_i \vee B_i) \mid A_i, B_i \in \mathfrak{J}^C \right\}$ is an ideal of \mathfrak{e}_6^C .

From the simplicity of \mathfrak{e}_6^C (Theorem 3.3.1) we have $\mathfrak{a} = \mathfrak{e}_6^C$.

3.5. Killing form of \mathfrak{e}_6^C

Definition. We define a symmetric inner product $(\phi_1, \phi_2)_6$ in \mathfrak{e}_6^C by

$$(\phi_1, \phi_2)_6 = (\delta_1, \delta_2)_4 + (T_1, T_2),$$

where $\phi_i = \delta_i + \tilde{T}_i$, $\delta_i \in \mathfrak{f}_4^C$, $T_i \in \mathfrak{J}_0^C$.

Lemma 3.5.1. (1) The inner product $(\phi_1, \phi_2)_6$ of \mathfrak{e}_6^C is \mathfrak{e}_6^C -adjoint invariant:

$$([\phi, \phi_1], \phi_2)_6 + (\phi_1, [\phi, \phi_2])_6 = 0, \quad \phi, \phi_i \in \mathfrak{e}_6^C.$$

(2) For $\phi \in \mathfrak{e}_6^C$, $A, B \in \mathfrak{J}^C$, we have

$$(\phi, A \vee B)_6 = (\phi A, B).$$

Proof. (1) For $\phi = \delta + \tilde{T}$, $\phi_i = \delta_i + \tilde{T}_i$, $\delta, \delta_i \in \mathfrak{f}_4^C$, $T, T_i \in \mathfrak{J}_0^C$, we have

$$([\phi, \phi_1], \phi_2)_6$$

$$= ([\delta + \tilde{T}, \delta_1 + \tilde{T}_1], \delta_2 + \tilde{T}_2)_6$$

$$\begin{aligned}
&= (([\delta, \delta_1] + [\tilde{T}, \tilde{T}_1]) + (\widetilde{\delta\tilde{T}_1 - \delta_1\tilde{T}}, \delta_2 + \tilde{T}_2)_6 \quad (\text{Theorem 3.2.2}) \\
&= ([\delta, \delta_1], \delta_2)_4 + ([\tilde{T}, \tilde{T}_1], \delta_2)_4 + (\delta T_1 - \delta_1 T, T_2) \\
&= -(\delta_1, [\delta, \delta_2])_4 + (\delta_2 T, T_1) + (\delta T_1, T_2) - (\delta_1 T, T_2) \quad (\text{Lemma 2.5.2}) \\
&= -(\delta_1, [\delta, \delta_2])_4 - (\delta_1 T, T_2) - (T_1, \delta T_2) + (T_1, \delta_2 T) \\
&= -(\delta_1 + \tilde{T}_1, [\delta, \delta_2] + [\tilde{T}, \tilde{T}_2]) + (\widetilde{\delta\tilde{T}_2 - \delta_2\tilde{T}})_6 \\
&= -(\delta_1 + \tilde{T}_1, [\delta + \tilde{T}, \delta_2 + \tilde{T}_2])_6 \\
&= -(\phi_1, [\phi, \phi_2])_6.
\end{aligned}$$

(2) For $\phi = \delta + \tilde{T}$, $\delta \in \mathfrak{f}_4^C$, $T \in \mathfrak{J}_0^C$, we have

$$\begin{aligned}
(\phi, A \vee B)_6 &= \left(\delta + \tilde{T}, [\tilde{A}, \tilde{B}] + \left(A \circ B - \frac{1}{3}(A, B)E \right)^\sim \right)_6 \\
&= (\delta, [\tilde{A}, \tilde{B}])_4 + \left(T, A \circ B - \frac{1}{3}(A, B)E \right) \\
&= (\delta A, B) + (\tilde{T}A, B) = ((\delta + \tilde{T})A, B) = (\phi A, B).
\end{aligned}$$

Lemma 3.5.2. *In \mathfrak{e}_6^C , we have*

$$\begin{aligned}
[(E_i - E_{i+1})^\sim, D] &= 0, \quad D \in \mathfrak{d}_4^C, & [(E_i - E_{i+1})^\sim, (E_j - E_{j+1})^\sim] &= 0, \\
[(E_i - E_{i+1})^\sim, \tilde{A}_i(a)] &= -\frac{1}{2}\tilde{F}_i(a), & [(E_i - E_{i+1})^\sim, \tilde{F}_i(a)] &= -\frac{1}{2}\tilde{A}_i(a), \\
[(E_i - E_{i+1})^\sim, \tilde{A}_{i+1}(a)] &= -\frac{1}{2}\tilde{F}_{i+1}(a), & [(E_i - E_{i+1})^\sim, \tilde{F}_{i+1}(a)] &= -\frac{1}{2}\tilde{A}_{i+1}(a), \\
[(E_i - E_{i+1})^\sim, \tilde{A}_{i+2}(a)] &= \tilde{F}_{i+2}(a), & [(E_i - E_{i+1})^\sim, \tilde{F}_{i+2}(a)] &= \tilde{A}_{i+2}(a).
\end{aligned}$$

Theorem 3.5.3. *The Killing form B_6 of the Lie algebra \mathfrak{e}_6^C is given by*

$$\begin{aligned}
B_6(\phi_1, \phi_2) &= 12(\phi_1, \phi_2)_6 \\
&= 12(\delta_1, \delta_2)_4 + 12(T_1, T_2) \\
&= \frac{4}{3}B_4(\delta_1, \delta_2) + 12(T_1, T_2) \\
&= 4\text{tr}(\phi_1\phi_2),
\end{aligned}$$

where $\phi_i = \delta_i + \tilde{T}_i$, $\delta_i \in \mathfrak{f}_4^C$, $T_i \in \mathfrak{J}_0^C$ and B_4 is the Killing form of \mathfrak{f}_4^C .

Proof. Since \mathfrak{e}_6^C is simple (Theorem 3.3.1), there exist $k, k' \in C$ such that

$$B_6(\phi_1, \phi_2) = k(\phi_1, \phi_2)_6 = k'\text{tr}(\phi_1\phi_2).$$

To determine these k, k' , let $\phi = \phi_1 = \phi_2 = (E_1 - E_2)^\sim$. Then we have

$$(\phi, \phi)_6 = ((E_1 - E_2)^\sim, (E_1 - E_2)^\sim)_6 = (E_1 - E_2, E_1 - E_2) = 2.$$

On the other hand, $(\text{ad}\phi)^2$ is calculated as follows.

$$[\phi, [\phi, \tilde{A}_1(e_i)]] = \left[\phi, -\frac{1}{2}\tilde{F}_1(e_i) \right] = \frac{1}{4}\tilde{A}_1(e_i),$$

$$\begin{aligned}
[\phi, [\phi, \tilde{A}_2(e_i)]] &= \left[\phi, -\frac{1}{2}\tilde{F}_2(e_i) \right] = \frac{1}{4}\tilde{A}_2(e_i), \\
[\phi, [\phi, \tilde{A}_3(e_i)]] &= [\phi, \tilde{F}_3(e_i)] = \tilde{A}_3(e_i), \\
[\phi, [\phi, \tilde{F}_1(e_i)]] &= \left[\phi, -\frac{1}{2}\tilde{A}_1(e_i) \right] = \frac{1}{4}\tilde{F}_1(e_i), \\
[\phi, [\phi, \tilde{F}_2(e_i)]] &= \left[\phi, -\frac{1}{2}\tilde{A}_2(e_i) \right] = \frac{1}{4}\tilde{F}_2(e_i), \\
[\phi, [\phi, \tilde{F}_3(e_i)]] &= [\phi, \tilde{A}_3(e_i)] = \tilde{F}_3(e_i), \\
\text{the others} &= 0.
\end{aligned}$$

Hence

$$B_6(\phi, \phi) = \text{tr}((\text{ad}\phi)^2) = \left(\frac{1}{4} \times 4 + 1 \times 2 \right) \times 8 = 24.$$

Therefore $k = 12$. Next, we will calculate $\text{tr}(\phi\phi)$ as follows.

$$\begin{aligned}
\phi\phi E_1 &= \phi E_1 = E_1, & \phi\phi \tilde{F}_1(e_i) &= -\frac{1}{2}\phi \tilde{F}_1(e_i) = \frac{1}{4}\tilde{F}_1(e_i), \\
\phi\phi E_2 &= -\phi E_2 = E_2, & \phi\phi \tilde{F}_2(e_i) &= \frac{1}{2}\phi \tilde{F}_2(e_i) = \frac{1}{4}\tilde{F}_2(e_i), \\
\phi\phi E_3 &= \phi 0 = 0, & \phi\phi \tilde{F}_3(e_i) &= \phi 0 = 0.
\end{aligned}$$

Hence

$$\text{tr}(\phi\phi) = 1 \times 2 + \frac{1}{4} \times 8 \times 2 = 6.$$

Therefore $k' = 4$.

Lemma 3.5.4. *The followings hold in \mathfrak{e}_6^C .*

$$(1) \quad A \vee (A \times A) = 0, \quad A \in \mathfrak{J}^C,$$

$$A \vee (B \times C) + B \vee (C \times A) + C \vee (A \times B) = 0, \quad A, B, C \in \mathfrak{J}^C.$$

$$(2) \quad \text{For } A \in \mathfrak{J}^C, A \neq 0, \text{ there exists } B \in \mathfrak{J}^C \text{ such that } A \vee B \neq 0.$$

Proof. (1) $(\phi, (A \times A) \vee A)_6 = (\phi(A \times A), A)$ (Lemma 3.5.1.(2))

$$= 2(\phi' A \times A, A) \text{ (Lemma 3.4.3.(1))} = 2(\phi' A, A \times A) = -2(A, \phi(A \times A))$$

$$= -2(\phi, (A \times A) \vee A)_6.$$

Hence $(\phi, (A \times A) \vee A)_6 = 0$ for all $\phi \in \mathfrak{e}_6^C$, so that $(A \times A) \vee A = 0$, that is, $A \vee (A \times A) = 0$ (Lemma 3.4.3.(2)). If we put $\lambda A + \mu B + \nu C$ in the place of A , then the result follows from the coefficient of $\lambda\mu\nu$.

(2) Assume that $A \vee B = 0$, that is $B \vee A = 0$ (Lemma 3.4.3.(2)) for all $B \in \mathfrak{J}^C$. Then for any $\phi \in \mathfrak{e}_6^C$, $0 = (\phi, B \vee A)_6 = (\phi B, A)$ (Lemma 3.5.1.(2)). Since $\mathfrak{e}_6^C \mathfrak{J}^C = \mathfrak{J}^C$ (Proposition 3.3.2.(3)), we have $(\mathfrak{J}^C, A) = 0$, so that $A = 0$.

3.6. Roots of \mathfrak{e}_6^C

Let

$$\mathfrak{M}^r = \{X \in M(3, \mathfrak{C}) \mid \text{all diagonal elements of } X \text{ are real}\},$$

and let $(\mathfrak{M}^r)^C$ be its complexification. In $(\mathfrak{M}^r)^C$, we define a multiplication $X \circ Y$ by

$$X \circ Y = \frac{1}{2}(XY + Y^*X^*),$$

where $X^* = {}^t\overline{X}$. For $\delta = (\delta_1, \delta_2, \delta_3) \in \mathfrak{d}_4^C$ satisfying the principle of triality $(\delta_1 x)y + x(\delta_2 y) = \overline{\delta_3(\overline{xy})}$, $x, y \in \mathfrak{C}^C$, we define a C -linear mapping $\delta : (\mathfrak{M}^r)^C \rightarrow (\mathfrak{M}^r)^C$ by

$$\delta \begin{pmatrix} \xi_1 & x_{12} & x_{13} \\ x_{21} & \xi_2 & x_{23} \\ x_{31} & x_{32} & \xi_3 \end{pmatrix} = \begin{pmatrix} 0 & \delta_3 x_{12} & \overline{\delta_2 \overline{x_{13}}} \\ \overline{\delta_3 \overline{x_{21}}} & 0 & \delta_1 x_{23} \\ \delta_2 x_{31} & \overline{\delta_1 \overline{x_{32}}} & 0 \end{pmatrix}.$$

Observe that this mapping δ is an extension of $\delta : \mathfrak{J}^C \rightarrow \mathfrak{J}^C$.

Lemma 3.6.1. *For $\delta \in \mathfrak{d}_4^C$, we have*

$$\delta(X \circ Y) = \delta X \circ Y + X \circ \delta Y, \quad X, Y \in (\mathfrak{M}^r)^C.$$

Proof. We use the following notations:

$$\begin{aligned} \sigma_{23} &= \delta_1, & \sigma_{31} &= \delta_2, & \sigma_{12} &= \delta_3, \\ \sigma_{32} &= \kappa \delta_1, & \sigma_{13} &= \kappa \delta_2, & \sigma_{21} &= \kappa \delta_3. \end{aligned}$$

The (i, i) -element of $\delta X \circ Y + Y \circ \delta Y$ (note that this is contained in C) is equal to

$$\begin{aligned} & R(\sum_{k=1}^3 ((\sigma_{ik} x_{ik}) y_{ki} + \overline{y_{ki}}(\overline{\sigma_{ik} x_{ik}})) + \sum_{k=1}^3 (x_{ik}(\sigma_{ki} y_{ki}) + (\overline{\sigma_{ki} y_{ki}}) \overline{x_{ik}})) \\ &= 2R(\sum_k ((\sigma_{ik} x_{ik}) y_{ki} + x_{ik}(\sigma_{ki} y_{ki}))) \\ &= 2 \sum_k ((\sigma_{ik} x_{ik}, \overline{y_{ki}}) + (x_{ik}, \overline{\sigma_{ki} y_{ki}})) \\ &= 2 \sum_k ((x_{ik}, -\sigma_{ik} \overline{y_{ki}}) + (x_{ik}, \sigma_{ik} \overline{y_{ki}})) = 0. \end{aligned}$$

The (i, j) -element of $\delta X \circ Y + Y \circ \delta Y$ ($i \neq j$) is equal to

$$\begin{aligned} & \sum_{k=1}^3 ((\sigma_{ik} x_{ik}) y_{kj} + \overline{y_{kj}}(\overline{\sigma_{ik} x_{ik}})) + \sum_{k=1}^3 (x_{ik}(\sigma_{kj} y_{kj}) + (\overline{\sigma_{kj} y_{kj}}) \overline{x_{ik}}) \\ &= \sum_k ((\sigma_{ik} x_{ik}) y_{kj} + y_{ik}(\sigma_{kj} x_{kj})) + \sum_k ((\sigma_{jk} x_{jk}) y_{ki} + x_{jk}(\sigma_{ki} y_{ki})) \end{aligned}$$

(If i, j, k are all distinct, from the principle of triality, we obtain

$$(\sigma_{ik} x_{ik}) y_{kj} + x_{ik}(\sigma_{kj} y_{kj}) = \sigma_{ij}(x_{ik} y_{kj}).$$

Even if $i = k \neq j$ or $i \neq k = j$, considering the fact that $\sigma_{ii} = 0$, we see that the formula above is also valid, since $x_{ii} \in C$)

$$\begin{aligned} &= \sum_k \sigma_{ij}(x_{ik} y_{kj}) + \sum_k \overline{\sigma_{ji}(x_{jk} y_{ki})} \\ &= \sum_k \sigma_{ij}(x_{ik} y_{kj}) + \sum_k \sigma_{ij}(\overline{x_{jk} y_{ki}}) = (i, j)\text{-element of } \delta(X \circ Y). \end{aligned}$$

For $T \in M(3, \mathfrak{C}^C)$, we define a C -linear mapping $\tilde{T} : \mathfrak{J}^C \rightarrow \mathfrak{J}^C$ by

$$\tilde{T}X = \frac{1}{2}(TX + XT^*), \quad \text{where } T^* = {}^t\overline{T}.$$

Proposition 3.6.2. For $T \in M(3, \mathfrak{C}^C)$, $\text{tr}(T) = 0$, we have $\tilde{T} \in \mathfrak{e}_6^C$.

Proof. We decompose $T = T_1 + T_2$, $T_1 = \frac{T + T^*}{2}$, $T_2 = \frac{T - T^*}{2}$. Then $\tilde{T}_1 \in \mathfrak{e}_6^C$ (Lemma 2.4.1.(1)) and $\tilde{T}_2 \in \mathfrak{f}_4^C$ (Proposition 2.3.6) $\subset \mathfrak{e}_6^C$. Therefore $\tilde{T} = \tilde{T}_1 + \tilde{T}_2 \in \mathfrak{e}_6^C$.

Lemma 3.6.3. (1) For $\delta \in \mathfrak{d}_4^C$ and $R \in (\mathfrak{M}^r)^C$, $\text{tr}(R) = 0$, we have

$$[\delta, \tilde{R}] = \tilde{\delta R}.$$

(2) For $H \in M(3, C)$, $\text{tr}(H) = 0$ and $T \in M(3, \mathfrak{C}^C)$, $\text{tr}(T) = 0$, we have

$$[\tilde{H}, \tilde{T}] = \frac{1}{2}[H, T]^\sim.$$

Proof. (1) $(\tilde{\delta R})X = \delta R \circ X = \delta(R \circ X) - R \circ \delta X$ (Lemma 3.6.1) $= \delta(\tilde{R}X) - \tilde{R}(\delta X) = [\delta, \tilde{R}]X$, $X \in \mathfrak{J}^C$. Hence, $\tilde{\delta R} = [\delta, \tilde{R}]$.

$$\begin{aligned} (2) \quad & [\tilde{H}, \tilde{T}]X = \tilde{H}\tilde{T}X - \tilde{T}\tilde{H}X \\ &= \frac{1}{2}(\tilde{H}(TX + XT^*) - \tilde{T}(HX + XH^*)) \\ &= \frac{1}{4}(HTX + XT^*H^* + HXT^* + TXH^* - THX - XH^*T^* - TXH^* - HXT^*) \\ & \text{(since } H \in M(3, C), \text{ products of matrices above are associative)} \\ &= \frac{1}{4}([H, T]X + X[H, T]^*) = \frac{1}{2}[H, T]^\sim X, \quad X \in \mathfrak{J}^C. \end{aligned}$$

Theorem 3.6.4. The rank of the Lie algebra \mathfrak{e}_6^C is 6. The roots of \mathfrak{e}_6^C relative to some Cartan subalgebra are given by

$$\begin{aligned} & \pm(\lambda_k - \lambda_l), \quad \pm(\lambda_k + \lambda_l), \quad 0 \leq k < l \leq 3, \\ & \pm\lambda_k \pm \frac{1}{2}(\mu_2 - \mu_3), \quad 0 \leq k \leq 3, \\ & \pm\frac{1}{2}(-\lambda_0 - \lambda_1 + \lambda_2 - \lambda_3) \pm \frac{1}{2}(\mu_3 - \mu_1), \\ & \pm\frac{1}{2}(\lambda_0 + \lambda_1 + \lambda_2 - \lambda_3) \pm \frac{1}{2}(\mu_3 - \mu_1), \\ & \pm\frac{1}{2}(-\lambda_0 + \lambda_1 + \lambda_2 + \lambda_3) \pm \frac{1}{2}(\mu_3 - \mu_1), \\ & \pm\frac{1}{2}(\lambda_0 - \lambda_1 + \lambda_2 + \lambda_3) \pm \frac{1}{2}(\mu_3 - \mu_1), \\ & \pm\frac{1}{2}(\lambda_0 - \lambda_1 + \lambda_2 - \lambda_3) \pm \frac{1}{2}(\mu_1 - \mu_2), \\ & \pm\frac{1}{2}(-\lambda_0 + \lambda_1 + \lambda_2 - \lambda_3) \pm \frac{1}{2}(\mu_1 - \mu_2), \\ & \pm\frac{1}{2}(\lambda_0 + \lambda_1 + \lambda_2 + \lambda_3) \pm \frac{1}{2}(\mu_1 - \mu_2), \\ & \pm\frac{1}{2}(-\lambda_0 - \lambda_1 + \lambda_2 + \lambda_3) \pm \frac{1}{2}(\mu_1 - \mu_2), \end{aligned}$$

with $\mu_1 + \mu_2 + \mu_3 = 0$.

Proof. We use the decomposition of Theorem 3.3.1.(2):

$$\mathfrak{e}_6^C = \mathfrak{f}_4^C \oplus \tilde{\mathfrak{f}}_0^C.$$

Let

$$\mathfrak{h} = \left\{ h = h_\delta + \tilde{H} \in \mathfrak{e}_6^C \left| \begin{array}{l} h_\delta = \sum_{k=0}^3 \lambda_k H_k = -\sum_{k=0}^3 \lambda_k i G_{k4+k}, \lambda_k \in C \\ H = \sum_{j=1}^3 \mu_j E_j, \mu_j \in C, \mu_1 + \mu_2 + \mu_3 = 0 \end{array} \right. \right\},$$

then \mathfrak{h} is an abelian subalgebra of \mathfrak{e}_6^C (it will be a Cartan subalgebra of \mathfrak{e}_6^C). That \mathfrak{h} is abelian is clear from

$$\begin{aligned} [h_\delta, h_{\delta'}] &= 0, \\ [h_\delta, \tilde{H}'] &= \widetilde{h_\delta H'} \text{ (Lemma 3.6.3.(1)) } = \tilde{0} = 0, \\ [\tilde{H}, \tilde{H}'] &= \frac{1}{2}[H, H']^\sim \text{ (Lemma 3.6.3.(2)) } = 0. \end{aligned}$$

I The roots $\pm\lambda_k \pm \lambda_l$ of $\mathfrak{d}_4^C (\subset \mathfrak{f}_4^C \subset \mathfrak{e}_6^C)$ are also roots of \mathfrak{e}_6^C . Indeed, let α be a root of \mathfrak{d}_4^C and $S \in \delta_4^C \subset \mathfrak{e}_6^C$ be an associated root vector. Then

$$\begin{aligned} [h, S] &= [h_\delta + \tilde{H}, S] = [h_\delta, S] - [S, \tilde{H}] \\ &= \alpha(h_\delta)S - \widetilde{S\tilde{H}} \text{ (Lemma 3.6.3.(1))} \\ &= \alpha(h_\delta)S = (\pm\lambda_k \pm \lambda_l)S. \end{aligned}$$

Hence $\pm\lambda_k \pm \lambda_l$ are roots of \mathfrak{e}_6^C .

II We denote $aE_{kl} \in M(3, \mathfrak{C}^C)$ by $F_{kl}(a)$: $F_{kl}(a) = aE_{kl}, a \in \mathfrak{C}^C, k \neq l$. For example, $F_{23}(a) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & a \\ 0 & 0 & 0 \end{pmatrix}$. Then, we have

$$\begin{aligned} [h, \tilde{F}_{23}(a)] &= [h_\delta, \tilde{F}_{23}(a)] + [\tilde{H}, \tilde{F}_{23}(a)] \\ &= (h_\delta F_{23}(a))^\sim + \frac{1}{2}[H, F_{23}(a)]^\sim \text{ (Lemma 3.6.3)} \\ &= \tilde{F}_{23}(h_\delta a) + \frac{1}{2}[H, F_{23}(a)]^\sim \\ &= \left(\lambda_k + \frac{1}{2}(\mu_2 - \mu_3) \right) \tilde{F}_{23}(a), \end{aligned}$$

where $a = e_k + ie_{4+k}$ (since $h_\delta a = h_\delta(e_k + ie_{4+k}) = \lambda_k(e_k + ie_{4+k}) = \lambda_k a$, and $[H, F_{23}(a)] = HF_{23}(a) - F_{23}(a)H = (\mu_2 - \mu_3)F_{23}(a)$). Hence $\lambda_k + \frac{1}{2}(\mu_2 - \mu_3)$ is a

root of \mathfrak{e}_6^C and $\tilde{F}_{23}(e_k + ie_{4+k})$ is its root vector. Similarly $-\lambda_k + \frac{1}{2}(\mu_2 - \mu_3)$ is a root of \mathfrak{e}_6^C and $\tilde{F}_{23}(e_k - ie_{4+k})$ is its root vector. Again, from

$$[h, \tilde{F}_{32}(a)] = \left(-\lambda_k + \frac{1}{2}(\mu_3 - \mu_2)\right) \tilde{F}_{32}(a), \quad a = e_k + ie_{4+k},$$

we see that $-\lambda_k + \frac{1}{2}(\mu_3 - \mu_2)$ is a root of \mathfrak{e}_6^C . Similarly $\lambda_k + \frac{1}{2}(\mu_3 - \mu_2)$ is a root of \mathfrak{e}_6^C and $\tilde{F}_{32}(e_k - ie_{4+k})$ is its root vector. Next, using the relation

$$\begin{aligned} [h, \tilde{F}_{31}(a)] &= (h_\delta F_{31}(a))^\sim + \frac{1}{2}[\tilde{H}, \tilde{F}_{31}(a)] = \tilde{F}_{31}((\nu h_\delta)a) + \frac{1}{2}(\mu_3 - \mu_1)\tilde{F}_{31}(a), \\ [h, \tilde{F}_{12}(a)] &= (h_\delta F_{12}(a))^\sim + \frac{1}{2}[\tilde{H}, \tilde{F}_{12}(a)] = \tilde{F}_{12}((\kappa\pi h_\delta)a) + \frac{1}{2}(\mu_1 - \mu_2)\tilde{F}_{12}(a), \end{aligned}$$

where

$$\begin{aligned} \nu h_\delta &= \frac{1}{2}(-\lambda_0 - \lambda_1 + \lambda_2 - \lambda_3)H_0 + \frac{1}{2}(\lambda_0 + \lambda_1 + \lambda_2 - \lambda_3)H_1 \\ &\quad + \frac{1}{2}(-\lambda_0 + \lambda_1 + \lambda_2 + \lambda_3)H_2 + \frac{1}{2}(\lambda_0 - \lambda_1 + \lambda_2 + \lambda_3)H_3, \\ \kappa\pi h_\delta &= \frac{1}{2}(-\lambda_0 + \lambda_1 - \lambda_2 + \lambda_3)H_0 + \frac{1}{2}(-\lambda_0 + \lambda_1 + \lambda_2 - \lambda_3)H_1 \\ &\quad + \frac{1}{2}(\lambda_0 + \lambda_1 + \lambda_2 + \lambda_3)H_2 + \frac{1}{2}(-\lambda_0 - \lambda_1 + \lambda_2 + \lambda_3)H_3 \end{aligned}$$

etc., we can obtain the remainders of roots.

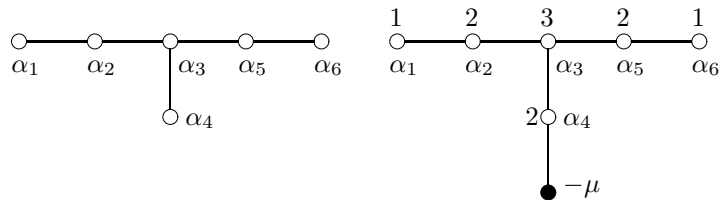
Theorem 3.6.5. *In the root system of Theorem 3.6.4,*

$$\begin{aligned} \alpha_1 &= \lambda_0 - \lambda_1, \quad \alpha_2 = \lambda_1 - \lambda_2, \quad \alpha_3 = \lambda_2 - \lambda_3, \\ \alpha_4 &= \lambda_3 + \frac{1}{2}(\mu_2 - \mu_3), \\ \alpha_5 &= \frac{1}{2}(-\lambda_0 - \lambda_1 - \lambda_2 + \lambda_3) + \frac{1}{2}(\mu_3 - \mu_1), \\ \alpha_6 &= \frac{1}{2}(\lambda_0 + \lambda_1 + \lambda_2 + \lambda_3) + \frac{1}{2}(\mu_1 - \mu_2) \end{aligned}$$

is a fundamental root system of the Lie algebra \mathfrak{e}_6^C and

$$\mu = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6$$

is the highest root. The Dynkin diagram and the extended Dynkin diagram of \mathfrak{e}_6^C are respectively given by



Proof. In the following, the notation $n_1 n_2 \cdots n_6$ denotes the root $n_1 \alpha_1 + n_2 \alpha_2 + \cdots + n_6 \alpha_6$. Now, all positive roots of \mathfrak{e}_6^C are represented by

$$\begin{array}{ll}
\lambda_0 - \lambda_1 = 1 & 0 & 0 & 0 & 0 & 0 & \lambda_0 + \lambda_1 = 1 & 2 & 2 & 1 & 1 & 1 \\
\lambda_0 - \lambda_2 = 1 & 1 & 0 & 0 & 0 & 0 & \lambda_0 + \lambda_2 = 1 & 1 & 2 & 1 & 1 & 1 \\
\lambda_0 - \lambda_3 = 1 & 1 & 1 & 0 & 0 & 0 & \lambda_0 + \lambda_3 = 1 & 1 & 1 & 1 & 1 & 1 \\
\lambda_1 - \lambda_2 = 0 & 1 & 0 & 0 & 0 & 0 & \lambda_1 + \lambda_2 = 0 & 1 & 2 & 1 & 1 & 1 \\
\lambda_1 - \lambda_3 = 0 & 1 & 1 & 0 & 0 & 0 & \lambda_1 + \lambda_3 = 0 & 1 & 1 & 1 & 1 & 1 \\
\lambda_2 - \lambda_3 = 0 & 0 & 1 & 0 & 0 & 0 & \lambda_2 + \lambda_3 = 0 & 0 & 1 & 1 & 1 & 1
\end{array}$$

$$\begin{array}{l}
\lambda_0 + \frac{1}{2}(\mu_2 - \mu_3) = 1 & 1 & 1 & 1 & 0 & 0 \\
\lambda_1 + \frac{1}{2}(\mu_2 - \mu_3) = 0 & 1 & 1 & 1 & 0 & 0 \\
\lambda_2 + \frac{1}{2}(\mu_2 - \mu_3) = 0 & 0 & 1 & 1 & 0 & 0 \\
\lambda_3 + \frac{1}{2}(\mu_2 - \mu_3) = 0 & 0 & 0 & 1 & 0 & 0 \\
\lambda_0 - \frac{1}{2}(\mu_2 - \mu_3) = 1 & 1 & 1 & 0 & 1 & 1 \\
\lambda_1 - \frac{1}{2}(\mu_2 - \mu_3) = 0 & 1 & 1 & 0 & 1 & 1 \\
\lambda_2 - \frac{1}{2}(\mu_2 - \mu_3) = 0 & 0 & 1 & 0 & 1 & 1 \\
\lambda_3 - \frac{1}{2}(\mu_2 - \mu_3) = 0 & 0 & 0 & 0 & 1 & 1
\end{array}$$

$$\begin{array}{l}
\frac{1}{2}(-\lambda_0 - \lambda_1 + \lambda_2 - \lambda_3) + \frac{1}{2}(\mu_3 - \mu_1) = 0 & 0 & 1 & 0 & 1 & 0 \\
\frac{1}{2}(\lambda_0 + \lambda_1 + \lambda_2 - \lambda_3) + \frac{1}{2}(\mu_3 - \mu_1) = 1 & 2 & 3 & 1 & 2 & 1 \\
\frac{1}{2}(-\lambda_0 + \lambda_1 + \lambda_2 + \lambda_3) + \frac{1}{2}(\mu_3 - \mu_1) = 0 & 1 & 2 & 1 & 2 & 1 \\
\frac{1}{2}(\lambda_0 - \lambda_1 + \lambda_2 + \lambda_3) + \frac{1}{2}(\mu_3 - \mu_1) = 1 & 1 & 2 & 1 & 2 & 1 \\
\frac{1}{2}(\lambda_0 + \lambda_1 - \lambda_2 + \lambda_3) + \frac{1}{2}(\mu_3 - \mu_1) = 1 & 2 & 2 & 1 & 2 & 1 \\
\frac{1}{2}(-\lambda_0 - \lambda_1 - \lambda_2 + \lambda_3) + \frac{1}{2}(\mu_3 - \mu_1) = 0 & 0 & 0 & 0 & 1 & 0 \\
\frac{1}{2}(\lambda_0 - \lambda_1 - \lambda_2 - \lambda_3) + \frac{1}{2}(\mu_3 - \mu_1) = 1 & 1 & 1 & 0 & 1 & 0 \\
\frac{1}{2}(-\lambda_0 + \lambda_1 - \lambda_2 - \lambda_3) + \frac{1}{2}(\mu_3 - \mu_1) = 0 & 1 & 1 & 0 & 1 & 0 \\
\frac{1}{2}(\lambda_0 - \lambda_1 + \lambda_2 - \lambda_3) - \frac{1}{2}(\mu_1 - \mu_2) = 1 & 1 & 2 & 1 & 1 & 0 \\
\frac{1}{2}(\lambda_0 - \lambda_1 - \lambda_2 + \lambda_3) - \frac{1}{2}(\mu_1 - \mu_2) = 1 & 1 & 1 & 1 & 1 & 0 \\
\frac{1}{2}(\lambda_0 + \lambda_1 + \lambda_2 + \lambda_3) + \frac{1}{2}(\mu_1 - \mu_2) = 0 & 0 & 0 & 0 & 0 & 1
\end{array}$$

$$\begin{aligned}
\frac{1}{2}(\lambda_0 + \lambda_1 - \lambda_2 - \lambda_3) - \frac{1}{2}(\mu_1 - \mu_2) &= 1 & 2 & 2 & 1 & 1 & 0 \\
\frac{1}{2}(-\lambda_0 + \lambda_1 - \lambda_2 + \lambda_3) - \frac{1}{2}(\mu_1 - \mu_2) &= 0 & 1 & 1 & 1 & 1 & 0 \\
\frac{1}{2}(-\lambda_0 + \lambda_1 + \lambda_2 - \lambda_3) - \frac{1}{2}(\mu_1 - \mu_2) &= 0 & 1 & 2 & 1 & 1 & 0 \\
\frac{1}{2}(\lambda_0 + \lambda_1 + \lambda_2 + \lambda_3) - \frac{1}{2}(\mu_1 - \mu_2) &= 1 & 2 & 3 & 2 & 2 & 1 \\
\frac{1}{2}(-\lambda_0 - \lambda_1 + \lambda_2 + \lambda_3) - \frac{1}{2}(\mu_1 - \mu_2) &= 0 & 0 & 1 & 1 & 1 & 0.
\end{aligned}$$

Hence $\Pi = \{\alpha_1, \alpha_2, \dots, \alpha_6\}$ is a fundamental root system of \mathfrak{e}_6^C . The real part $\mathfrak{h}_{\mathbf{R}}$ of \mathfrak{h} is

$$\mathfrak{h}_{\mathbf{R}} = \left\{ \sum_{k=0}^3 \lambda_k H_k + \left(\sum_{j=1}^3 \mu_j E_j \right)^{\sim} \mid \lambda_k, \mu_j \in \mathbf{R}, \mu_1 + \mu_2 + \mu_3 = 0 \right\}.$$

The Killing form B_6 of \mathfrak{e}_6^C on $\mathfrak{h}_{\mathbf{R}}$ is given by

$$B_6(h, h') = 12 \left(2 \sum_{k=0}^3 \lambda_k \lambda_k' + \sum_{j=1}^3 \mu_j \mu_j' \right)$$

for $h = \sum_{k=0}^3 \lambda_k H_k + \left(\sum_{j=1}^3 \mu_j E_j \right)^{\sim}$, $h' = \sum_{k=0}^3 \lambda_k' H_k + \left(\sum_{j=1}^3 \mu_j' E_j \right)^{\sim} \in \mathfrak{h}_{\mathbf{R}}$. Indeed, from Theorem 3.5.3, we have

$$\begin{aligned}
B_6(h, h') &= \frac{4}{3} B_4 \left(\sum_{k=0}^3 \lambda_k H_k, \sum_{k=0}^3 \lambda_k' H_k \right) + 12 \left(\sum_{j=1}^3 \mu_j E_j, \sum_{j=1}^3 \mu_j' E_j \right) \\
&= \frac{4}{3} 18 \sum_{k=0}^3 \lambda_k \lambda_k' + 12 \sum_{j=1}^3 \mu_j \mu_j' \quad (\text{Theorem 2.6.2}) \\
&= 12 \left(2 \sum_{k=0}^3 \lambda_k \lambda_k' + \sum_{j=1}^3 \mu_j \mu_j' \right).
\end{aligned}$$

Now, the canonical elements $H_{\alpha_i} \in \mathfrak{h}_{\mathbf{R}}$ corresponding to α_i ($B_6(H_{\alpha}, H) = \alpha(H)$, $H \in \mathfrak{h}_{\mathbf{R}}$) are determined as follows.

$$\begin{aligned}
H_{\alpha_1} &= \frac{1}{24}(H_0 - H_1), & H_{\alpha_2} &= \frac{1}{24}(H_1 - H_2), & H_{\alpha_3} &= \frac{1}{24}(H_2 - H_3), \\
H_{\alpha_4} &= \frac{1}{24}(H_3 + (E_2 - E_3)^{\sim}), \\
H_{\alpha_5} &= \frac{1}{48}((-H_0 - H_1 - H_2 + H_3) + 2(E_3 - E_1)^{\sim}), \\
H_{\alpha_6} &= \frac{1}{48}((H_0 + H_1 + H_2 + H_3) + 2(E_1 - E_2)^{\sim}).
\end{aligned}$$

Therefore, we have

$$(\alpha_1, \alpha_1) = B_6(H_{\alpha_1}, H_{\alpha_1}) = 24 \frac{1}{24} \frac{1}{24} 2 = \frac{1}{12},$$

and the other inner products are similarly calculated. Hence, the inner product induced by the Killing form B_6 between $\alpha_1, \alpha_2, \dots, \alpha_6$ and $-\mu$ are given by

$$\begin{aligned} (\alpha_i, \alpha_i) &= \frac{1}{12}, \quad i = 1, 2, 3, 4, 5, 6, \\ (\alpha_1, \alpha_2) &= (\alpha_2, \alpha_3) = (\alpha_3, \alpha_4) = (\alpha_3, \alpha_5) = (\alpha_5, \alpha_6) = -\frac{1}{24}, \\ (\alpha_i, \alpha_j) &= 0, \quad \text{otherwise,} \\ (-\mu, -\mu) &= \frac{1}{12}, \quad (-\mu, \alpha_4) = -\frac{1}{24}, \quad (-\mu, \alpha_i) = 0, \quad i = 1, 2, 3, 5, 6, \end{aligned}$$

using them, we can draw the Dynkin diagram and the extended Dynkin diagram of \mathfrak{e}_6^C .

According to Borel-Siebenthal theory, the Lie algebra \mathfrak{e}_6 has three subalgebras as maximal subalgebras with the maximal rank 6.

(1) The first one is a subalgebra of type $T \oplus D_5$ which is obtained as the fixed points of an involution σ of \mathfrak{e}_6 .

(2) The second one is a subalgebra of type $C_1 \oplus A_5$ which is obtained as the fixed points of an involution γ of \mathfrak{e}_6 .

(3) The third one is a subalgebra of type $A_2 \oplus A_2 \oplus A_2$ which is obtained as the fixed points of an automorphism w of order 3 of \mathfrak{e}_6 .

The Lie algebra \mathfrak{e}_6 has furthermore two outer involutions $\tau, \tau\gamma$. The subalgebra obtained as the fixed points of τ is type F_4 and the subalgebra obtained as the fixed points of $\tau\gamma$ is type C_4 .

These subalgebras will be realized as subgroups of the group E_6 in Theorems 3.10.7, 3.11.4, 3.13.5, 3.7.1 and 3.12.2, respectively.

3.7. Involution τ and subgroup F_4 of E_6

We shall study the following subgroup $(E_6)^\tau$ of E_6 :

$$\begin{aligned} (E_6)^\tau &= \{\alpha \in E_6 \mid \tau\alpha = \alpha\tau\} \\ &= \{\alpha \in E_6 \mid \lambda(\alpha) = \alpha\} = (E_6)^\lambda. \end{aligned}$$

If $\alpha \in E_6$ satisfies $\tau\alpha = \alpha\tau$, then $(\alpha X, \alpha Y) = \langle \tau\alpha X, \alpha Y \rangle = \langle \alpha\tau X, \alpha Y \rangle = \langle \tau X, Y \rangle = (X, Y)$ and vice versa. Further, for $\alpha \in E_6$, the conditions $(\alpha X, \alpha Y) = (X, Y)$ and $\alpha E = E$ are equivalent (Lemma 2.2.4). Hence $(E_6)^\tau$ can be also defined by

$$\begin{aligned} (E_6)^\tau &= \{\alpha \in E_6 \mid (\alpha X, \alpha Y) = (X, Y), X, Y \in \mathfrak{J}^C\} \\ &= \{\alpha \in E_6 \mid \alpha E = E\} = (E_6)_E. \end{aligned}$$

Theorem 3.7.1. $(E_6)^\tau = (E_6)_E \cong F_4$.

(From now on, we identify these groups).

Proof. We shall show $(E_6)^\tau \cong F_4$. Let $\alpha \in (E_6)^\tau$. Then, for $X \in \mathfrak{J}$ we have $\tau\alpha X = \alpha\tau X = \alpha X$, so that $\alpha X \in \mathfrak{J}$. Hence α induces an \mathbf{R} -linear transformation of \mathfrak{J} . Therefore, the restriction $\alpha|_{\mathfrak{J}}$ of α to \mathfrak{J} belongs to the group

$$\begin{aligned} F_4 &= \{ \alpha \in \text{Iso}_{\mathbf{R}}(\mathfrak{J}) \mid \det(\alpha X) = \det X, (\alpha X, \alpha Y) = (X, Y) \} \\ &= \{ \alpha \in \text{Iso}_{\mathbf{R}}(\mathfrak{J}) \mid \alpha(X \times Y) = \alpha X \times \alpha Y \}. \end{aligned}$$

Conversely, for $\alpha \in F_4$, its complexification $\alpha^C : \mathfrak{J}^C \rightarrow \mathfrak{J}^C$, $\alpha^C(X_1 + iX_2) = \alpha X_1 + i\alpha X_2$ belongs to $(E_6)^\tau$. Therefore the correspondence $F_4 \ni \alpha \rightarrow \alpha^C \in (E_6)^\tau$ gives an isomorphism between F_4 and $(E_6)^\tau$.

3.8. Connectedness of E_6

We denote by $(E_6)_0$ the connected component of E_6 containing the identity 1.

Lemma 3.8.1. (1) For $t \in \mathbf{R}$, if we define a mapping $\alpha_{12}(t) : \mathfrak{J}^C \rightarrow \mathfrak{J}^C$ by

$$\alpha_{12}(t) \begin{pmatrix} \xi_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & \xi_2 & x_1 \\ x_2 & \bar{x}_1 & \xi_3 \end{pmatrix} = \begin{pmatrix} e^{it}\xi_1 & x_3 & e^{it/2}\bar{x}_2 \\ \bar{x}_3 & e^{-it}\xi_2 & e^{-it/2}x_1 \\ e^{it/2}x_2 & e^{-it/2}\bar{x}_1 & \xi_3 \end{pmatrix},$$

then $\alpha_{12}(t) \in (E_6)_0$. Similarly we can define $\alpha_{13}(t), \alpha_{23}(t) \in (E_6)_0$.

(2) For $a \in \mathfrak{C}$, if we define a mapping $\alpha_1(a) : \mathfrak{J}^C \rightarrow \mathfrak{J}^C$ by $\alpha_1(a)X(\xi, x) = Y(\eta, y)$, where

$$\begin{cases} \eta_1 = \xi_1 \\ \eta_2 = \frac{\xi_2 - \xi_3}{2} + \frac{\xi_2 + \xi_3}{2} \cos |a| + i \frac{(a, x_1)}{|a|} \sin |a| \\ \eta_3 = -\frac{\xi_2 - \xi_3}{2} + \frac{\xi_2 + \xi_3}{2} \cos |a| + i \frac{(a, x_1)}{|a|} \sin |a|, \\ y_1 = x_1 + i \frac{(\xi_2 + \xi_3)a}{2|a|} \sin |a| - \frac{2(a, x_1)a}{|a|^2} \sin^2 \frac{|a|}{2} \\ y_2 = x_2 \cos \frac{|a|}{2} + i \frac{\bar{x}_3 a}{|a|} \sin \frac{|a|}{2} \\ y_3 = x_3 \cos \frac{|a|}{2} + i \frac{\bar{a} x_2}{|a|} \sin \frac{|a|}{2} \end{cases}$$

(if $a = 0$, then $\frac{\sin |a|}{|a|}$ means 1), then $\alpha_1(a) \in (E_6)_0$.

Proof. (1) For $E_1 - E_2 \in \mathfrak{J}_0$, we have $i(E_1 - E_2)^\sim \in \mathfrak{e}_6$ (Theorem 3.2.4.(2)) and $\alpha_{12}(t) = \exp it(E_1 - E_2)^\sim$. Hence $\alpha_{12}(t) \in (E_6)_0$.

(2) For $F_1(a) \in \mathfrak{J}_0$, we have $i\tilde{F}_1(a) \in \mathfrak{e}_6$ (Theorem 3.2.4.(2)) and $\alpha_1(a) = \exp i\tilde{F}_1(a)$. Hence $\alpha_1(a) \in (E_6)_0$.

Proposition 3.8.2. Any element $X \in \mathfrak{J}^C$ can be transformed to a diagonal form by some element $\alpha \in (E_6)_0$:

$$\alpha X = \begin{pmatrix} \xi_1 & 0 & 0 \\ 0 & \xi_2 & 0 \\ 0 & 0 & \xi_3 \end{pmatrix}, \quad \xi_i \in C.$$

Moreover, we can choose $\alpha \in (E_6)_0$ so that two of ξ_1, ξ_2, ξ_3 are non-negative real numbers.

Proof. (Hereafter, we use the notation $|\xi|$ instead of $\sqrt{\xi(\tau\xi)}$ for $\xi \in \mathbf{R}^C = C$). For a given $X \in \mathfrak{J}^C$, consider a space $\mathfrak{X} = \{\alpha X \mid \alpha \in (E_6)_0\}$. Since $(E_6)_0$ is compact (Theorem 3.1.1), \mathfrak{X} is also compact. Let $|\xi_1|^2 + |\xi_2|^2 + |\xi_3|^2$ be the maximal value of all $|\eta_1|^2 + |\eta_2|^2 + |\eta_3|^2$ for $Y = Y(\eta, y) \in \mathfrak{X}$ and let $X_0 = X(\xi, x)$ be an element of \mathfrak{X} which attains its maximal value. Then X_0 is of diagonal form. Certainly, suppose that X_0 is not of diagonal form, for example, the 2×3 entry x_1 of X_0 is non-zero:

$$0 \neq x_1 = p + iq, \quad p, q \in \mathfrak{C}.$$

It is sufficient to prove in the case that ξ_2, ξ_3 are real numbers (otherwise we can apply some $\alpha_{12}(t_1)$ and $\alpha_{13}(t_2)$ of Lemma 3.8.1.(1)).

(1) Case $q \neq 0$. Let $a(t) = \frac{q}{|q|}t$, $t > 0$ and construct $\alpha_1(a(t)) \in (E_6)_0$ of Lemma 3.8.1.(2). Since $|a(t)| = t$ and $i \frac{(a(t), x_1)}{|a(t)|} = i\nu - |q|$, where $\nu = \left(\frac{q}{|q|}, p\right)$, for $Y(\eta(t), y(t)) = \alpha_1(a(t))X_0 \in \mathfrak{X}$, we have

$$\begin{aligned} & |\eta_1(t)|^2 + |\eta_2(t)|^2 + |\eta_3(t)|^2 \\ &= |\xi_1|^2 + \left| \frac{\xi_2 - \xi_3}{2} + \frac{\xi_2 + \xi_3}{2} \cos t - |q| \sin t + i\nu \sin t \right|^2 \\ & \quad + \left| -\frac{\xi_2 - \xi_3}{2} + \frac{\xi_2 + \xi_3}{2} \cos t - |q| \sin t + i\nu \sin t \right|^2 \\ &= |\xi_1|^2 + 2\left(\frac{\xi_2 - \xi_3}{2}\right)^2 + 2\left(\frac{\xi_2 + \xi_3}{2} \cos t - |q| \sin t\right)^2 + 2\nu^2 \sin^2 t \\ &= |\xi_1|^2 + 2\left(\frac{\xi_2 - \xi_3}{2}\right)^2 + 2\left(\left(\frac{\xi_2 + \xi_3}{2}\right)^2 + |q|^2\right) \sin^2(t + t_0) + 2\nu^2 \sin^2 t \\ &\leq |\xi_1|^2 + |\xi_2|^2 + |\xi_3|^2 + 2|q|^2 + 2\nu^2 \cos^2 t_0 \quad (\text{for some } t_0 \in \mathbf{R}) \end{aligned}$$

which is the maximal value and attains at some $t > 0$. This contradicts the maximum of $|\xi_1|^2 + |\xi_2|^2 + |\xi_3|^2$. Hence $q = 0$.

(2) Case $p \neq 0$. Let $a(t) = \frac{p}{|p|}t$, $t > 0$ and construct $\beta_1(a(t)) \in F_4 \subset (E_6)_0$ of Lemma 2.8.1. For $Y(\eta(t), y(t)) = \beta_1(a(t))X_0 \in \mathfrak{X}$, the maximal value of $|\eta_1(t)|^2 + |\eta_2(t)|^2 + |\eta_3(t)|^2$ is $|\xi_1|^2 + |\xi_2|^2 + |\xi_3|^2 + 2|p|^2$ (the calculation is the same as Proposition 2.8.2) which contradicts the maximum of $|\xi_1|^2 + |\xi_2|^2 + |\xi_3|^2$. Hence $p = 0$. Consequently, we have $x_1 = 0$. $x_2 = x_3 = 0$ can be similarly proved constructing $\alpha_2(a)$, $\alpha_3(a) \in (E_6)_0$ analogous to $\alpha_1(a)$ of Lemma 3.8.1.(2). Hence X_0 is of diagonal form.

The space EIV , called the symmetric space of type EIV , is defined by

$$EIV = \{X \in \mathfrak{J}^C \mid \det X = 1, \langle X, X \rangle = 3\}.$$

Theorem 3.8.3. $E_6/F_4 \simeq EIV$.

In particular, E_6 is connected.

Proof. For $\alpha \in E_6$ and $X \in EIV$, we have $\alpha X \in EIV$. Hence E_6 acts on EIV . We shall prove that the group $(E_6)_0$ acts transitively on EIV . To prove this, it is sufficient to show that any element $X \in EIV$ can be transformed to $E \in EIV$ by some $\alpha \in (E_6)_0$. Now, $X \in EIV \subset \mathfrak{J}^C$ can be transformed to a diagonal form by $\alpha \in (E_6)_0$:

$$\alpha X = \begin{pmatrix} \xi_1 & 0 & 0 \\ 0 & \xi_2 & 0 \\ 0 & 0 & \xi_3 \end{pmatrix}, \quad \xi_1 \in C, \xi_2 \geq 0, \xi_3 \geq 0$$

(Proposition 3.8.2). From the condition $X \in EIV$, we have

$$\begin{aligned} \xi_1 \xi_2 \xi_3 &= \det(\alpha X) = \det X = 1, \quad (\text{hence } \xi_i > 0, i = 1, 2, 3), \\ \xi_1^2 + \xi_2^2 + \xi_3^2 &= \langle \alpha X, \alpha X \rangle = \langle X, X \rangle = 3. \end{aligned}$$

This implies that $\xi_1 = \xi_2 = \xi_3 = 1$. Certainly, from $0 \leq (\xi_2 - \xi_3)^2 = \xi_2^2 + \xi_3^2 - 2\xi_2\xi_3 = 3 - \xi_1^2 - \frac{2}{\xi_1} = -\frac{\xi_1^3 - 3\xi_1 + 2}{\xi_1} = -\frac{(\xi_1 - 1)^2(\xi_1 + 2)}{\xi_1} \leq 0$, we have $\xi_1 = 1$. Similarly $\xi_2 = \xi_3 = 1$ are obtained. Hence $\alpha X = E$, which shows the transitivity of $(E_6)_0$. Since we have $EIV = (E_6)_0 E$, EIV is connected. Now, the group E_6 acts transitively on EIV and the isotropy subgroup of E_6 at $E \in EIV$ is F_4 (Theorem 3.7.1). Thus we have the homeomorphism $E_6/F_4 \simeq EIV$. Finally, the connectedness of E_6 follows from the connectedness of EIV and F_4 .

3.9. Center $z(E_6)$ of E_6

Theorem 3.9.1. *The center $z(E_6)$ of the group E_6 is isomorphic to the cyclic group of order 3:*

$$z(E_6) = \{1, \omega 1, \omega^2 1\}, \quad \omega = -\frac{1}{2} + \frac{\sqrt{3}}{2}i \in C.$$

Proof. Let $\alpha \in z(E_6)$. From the commutativity with $\beta \in F_4 \subset E_6$, we have $\beta \alpha E = \alpha \beta E = \alpha E$. Let denote $\alpha E = Y = Y(\eta, y) \in \mathfrak{J}^C$, then we have

$$\beta Y = Y, \quad \text{for all } \beta \in F_4.$$

We choose $\beta \in F_4$ such that

$$\beta X = T X T^{-1}, \quad X \in \mathfrak{J}^C,$$

where $T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$, $\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \in SO(3)$. Then we have $y_1 = y_2 = y_3 = 0$ and $\eta_1 = \eta_2 = \eta_3 (= \omega)$, that is,

$$\alpha E = Y = \omega E, \quad \omega \in C,$$

and $\omega^3 = \det(\alpha E) = \det E = 1$. Since it is easy to verify that $\omega 1 \in z(E_6)$, we have $\omega^{-1}\alpha \in z(E_6)$ and $\omega^{-1}\alpha E = E$, and so $\omega^{-1}\alpha \in z(F_4)$ (Theorem 3.7.1). Since $z(F_4) = \{1\}$ (Theorem 2.10.1), we have $\omega^{-1}\alpha = 1$, that is, $\alpha = \omega 1$. This completes the proof.

According to the general theory of compact Lie groups, it is known that the center of the simply connected compact simple Lie group of type E_6 is the cyclic group of order 3. Hence the group E_6 has to be simply connected. Thus we have the following theorem.

Theorem 3.9.2. $E_6 = \{\alpha \in \text{Iso}_C(\mathfrak{J}^C) \mid \det(\alpha X) = \det X, \langle \alpha X, \alpha Y \rangle = \langle X, Y \rangle\}$ is a simply connected compact Lie group of type E_6 .

3.10. Involution σ and subgroup $(U(1) \times \text{Spin}(10))/\mathbf{Z}_4$ of E_6

Let the C -linear mapping $\sigma : \mathfrak{J}^C \rightarrow \mathfrak{J}^C$ be the complexification of $\sigma \in F_4$ of Section 2.9. Then $\sigma \in E_6$ and $\sigma^2 = 1$.

We shall study the following subgroup $(E_6)^\sigma$ of E_6 :

$$(E_6)^\sigma = \{\alpha \in E_6 \mid \sigma\alpha = \alpha\sigma\}.$$

To this end, we consider the C -subspaces $(\mathfrak{J}^C)_\sigma$ and $(\mathfrak{J}^C)_{-\sigma}$ of \mathfrak{J}^C , which are the eigenspaces of σ :

$$\begin{aligned} (\mathfrak{J}^C)_\sigma &= \{X \in \mathfrak{J}^C \mid \sigma X = X\} \\ &= \{X \in \mathfrak{J}^C \mid 4E_1 \times (E_1 \times X) = X\} \oplus \mathfrak{E}_1^C, \\ (\mathfrak{J}^C)_{-\sigma} &= \{X \in \mathfrak{J}^C \mid \sigma X = -X\} \\ &= \{X \in \mathfrak{J}^C \mid E_1 \times X = 0, \langle E_1, X \rangle = 0\}, \end{aligned}$$

where $\mathfrak{E}_1^C = \{\xi E_1 \mid \xi \in C\}$. Then $\mathfrak{J}^C = (\mathfrak{J}^C)_\sigma \oplus (\mathfrak{J}^C)_{-\sigma}$ (which is the complexification of $\mathfrak{J} = \mathfrak{J}_\sigma \oplus \mathfrak{J}_{-\sigma}$ in Section 2.9), and $(\mathfrak{J}^C)_\sigma, (\mathfrak{J}^C)_{-\sigma}$ are invariant under the action of $(E_6)^\sigma$.

Lemma 3.10.1. For $\alpha \in (E_6)^\sigma$, there exists $\xi \in C$ such that

$$\alpha E_1 = \xi E_1, \quad (\tau\xi)\xi = 1.$$

Proof. By the analogous proof to that of Theorem 2.9.1, we see that

$$\alpha E_2, \alpha E_3 \in \mathfrak{J}(2, \mathfrak{C}^C).$$

Indeed, we have

$$\begin{aligned}
\alpha E_2 &= \alpha(-F_2(1) \times F_2(1)) = -\tau\alpha\tau F_2(1) \times \tau\alpha\tau F_2(1) \\
&= -(F_2(x_2) + F_3(x_3)) \times (F_2(x_2) + F_3(x_3)) \quad (\text{for some } x_2, x_3 \in \mathfrak{C}^C) \\
&= (x_2, x_2)E_2 + (x_3, x_3)E_3 - F_1(\overline{x_2x_3}) \in \mathfrak{J}(2, \mathfrak{C}^C).
\end{aligned}$$

Next, we shall show

$$\alpha E_1 \notin \mathfrak{J}(2, \mathfrak{C}^C).$$

Suppose that $\alpha E_1 \in \mathfrak{J}(2, \mathfrak{C}^C)$. Then $\alpha E = \alpha E_1 + \alpha E_2 + \alpha E_3 \in \mathfrak{J}(2, \mathfrak{C}^C)$, so we can put $\alpha E = \xi_2 E_2 + \xi_3 E_3 + F_1(x_1)$, $\xi_2, \xi_3 \in C$, $x_1 \in \mathfrak{C}^C$. Hence

$$\begin{aligned}
\xi_2 E_2 + \xi_3 E_3 + F_1(x_1) &= \alpha E = \alpha(E \times E) = \tau\alpha\tau E \times \tau\alpha\tau E \\
&= \tau(\xi_2 E_2 + \xi_3 E_3 + F_1(x_1)) \times \tau(\xi_2 E_2 + \xi_3 E_3 + F_1(x_1)) \\
&= (\tau\xi_2\tau\xi_3 - (\tau x_1, \tau x_1))E_1.
\end{aligned}$$

This implies that $\xi_2 = \xi_3 = x_1 = 0$. Hence $\alpha E = 0$, which is a contradiction. Therefore αE_1 is of the form

$$\alpha E_1 = \xi E_1 + \xi_2 E_2 + \xi_3 E_3 + F_1(x_1), \quad \xi \neq 0.$$

From $\alpha E_1 \times \alpha E_1 = \tau\alpha\tau(E_1 \times E_1) = 0$, we have

$$\begin{aligned}
0 &= (\xi E_1 + \xi_2 E_2 + \xi_3 E_3 + F_1(x_1)) \times (\xi E_1 + \xi_2 E_2 + \xi_3 E_3 + F_1(x_1)) \\
&= (\xi_2\xi_3 - (x_1, x_1))E_1 + \xi\xi_3 E_2 + \xi\xi_2 E_3 - \xi F_1(x_1).
\end{aligned}$$

This implies that $\xi_2 = \xi_3 = x_1 = 0$ and so

$$\alpha E_1 = \xi E_1, \quad \xi \neq 0.$$

Finally, from

$$1 = \langle E_1, E_1 \rangle = \langle \alpha E_1, \alpha E_1 \rangle = \langle \xi E_1, \xi E_1 \rangle = (\tau\xi)\xi \langle E_1, E_1 \rangle = (\tau\xi)\xi,$$

we have $(\tau\xi)\xi = 1$.

In order to investigate the group $(E_6)^\sigma$, we consider the following subgroup $(E_6)_{E_1}$ of E_6 :

$$(E_6)_{E_1} = \{\alpha \in E_6 \mid \alpha E_1 = E_1\}.$$

Lemma 3.10.2. $(E_6)_{E_1}$ is a subgroup of $(E_6)^\sigma$: $(E_6)_{E_1} \subset (E_6)^\sigma$.

Proof. Since $(\mathfrak{J}^C)_\sigma = \{X \in \mathfrak{J}^C \mid 4E_1 \times (E_1 \times X) = X\} \oplus \mathfrak{E}_1^C$ and $(\mathfrak{J}^C)_{-\sigma} = \{X \in \mathfrak{J}^C \mid E_1 \times X = 0, \langle E_1, X \rangle = 0\}$, these spaces are seen to be invariant under the action of $(E_6)_{E_1}$. Hence for $\alpha \in (E_6)_{E_1}$, we have $\sigma\alpha = \alpha\sigma$ (the proof is the same as that of Theorem 2.9.1). Thus we have $\alpha \in (E_6)^\sigma$.

We define a 10 dimensional \mathbf{R} -vector space V^{10} by

$$V^{10} = \{X \in \mathfrak{J}^C \mid 2E_1 \times X = -\tau X\} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & \xi & x \\ 0 & \bar{x} & -\tau\xi \end{pmatrix} \mid \xi \in C, x \in \mathfrak{C} \right\}.$$

Proposition 3.10.3. $(E_6)_{E_1}/Spin(9) \simeq S^9$.

In particular, the group $(E_6)_{E_1}$ is connected.

Proof. $S^9 = \{X \in V^{10} \mid \langle X, X \rangle = 2\}$ is a 9 dimensional sphere. For $\alpha \in (E_6)_{E_1}$ and $X \in S^9$, we have $\alpha X \in S^9$. Indeed,

$$\begin{aligned} 2E_1 \times \alpha X &= 2\alpha E_1 \times \alpha X = 2\tau\alpha\tau(E_1 \times X) = \tau\alpha\tau(-\tau X) = -\tau(\alpha X), \\ \langle \alpha X, \alpha X \rangle &= \langle X, X \rangle = 2. \end{aligned}$$

Hence the group $(E_6)_{E_1}$ acts on S^9 . We shall prove that the action is transitive. To prove this, it is sufficient to show that any element $X \in S^9$ can be transformed to $i(E_2 + E_3) \in S^9$ by some $\alpha \in (E_6)_{E_1}$. Now, for a given $X \in S^9$, we can choose $\alpha_{23}(t_0)$ of Lemma 3.8.1.(1) such that

$$\alpha_{23}(t_0)X \in S^8 = \{X \in V^9 \mid \langle X, X \rangle = 2\}$$

where $V^9 = \{X \in V^{10} \mid \tau X = X\}$. (Note that $\alpha_{23}(t_0) \in (E_6)_{E_1}$ because $\alpha_{23}(t_0)E_1 = E_1$). Since the group $Spin(9) = (F_4)_{E_1} \subset (E_6)_{E_1}$ acts transitively on S^8 (Proposition 2.7.3), there exists $\beta \in Spin(9)$ such that

$$\beta\alpha_{23}(t_0)X = E_2 - E_3 \in S^8.$$

By applying $\alpha_{23}(\pi/2) \in (E_6)_{E_1}$ of Lemma 3.8.1.(1), we get

$$\alpha_{23}(\pi/2)\beta\alpha_{23}(t_0)X = i(E_2 + E_3).$$

This shows the transitivity. The isotropy subgroup of $(E_6)_{E_1}$ at $i(E_2 + E_3) \in S^9$ is $Spin(9)$. Indeed, if $\alpha \in (E_6)_{E_1}$ satisfies $\alpha(i(E_2 + E_3)) = i(E_2 + E_3)$, then $\alpha E = \alpha E_1 + \alpha(E_2 + E_3) = E_1 + (E_2 + E_3) = E$, so that $\alpha \in F_4$ and hence $\alpha \in (F_4)_{E_1} = Spin(9)$. Conversely $\alpha \in Spin(9)$ satisfies $\alpha(i(E_2 + E_3)) = i(E_2 + E_3)$. Thus we have the homeomorphism $(E_6)_{E_1}/Spin(9) \simeq S^9$.

Theorem 3.10.4. $(E_6)_{E_1} \cong Spin(10)$.

(From now on, we identify these groups).

Proof. Analogously to Theorem 2.7.4, we can define a homomorphism

$$p : (E_6)_{E_1} \rightarrow SO(10) = SO(V^{10})$$

by $p(\alpha) = \alpha|V^{10}$. The restriction p' of $p : (E_6)_{E_1} \rightarrow SO(10)$ to $(F_4)_{E_1}$ coincides with the homomorphism $p' : Spin(9) \rightarrow SO(9)$ of Theorem 2.7.4. In particular, $p' : Spin(9) \rightarrow SO(9)$ is onto. Hence, from the following commutative diagram

$$\begin{array}{ccccccc}
1 & \longrightarrow & Spin(9) & \longrightarrow & (E_6)_{E_1} & \longrightarrow & S^9 \longrightarrow * \\
& & \downarrow p' & & \downarrow p & & \downarrow = \\
1 & \longrightarrow & SO(9) & \longrightarrow & SO(10) & \longrightarrow & S^9 \longrightarrow *
\end{array}$$

we see that $p : (E_6)_{E_1} \rightarrow SO(10)$ is onto by the five lemma. We also have $\text{Ker } p = \{1, \sigma\}$. Indeed, $\alpha \in \text{Ker } p$ leaves $E_2 - E_3$ and $i(E_2 + E_3)$ invariant, so that $\alpha E_i = E_i$ for $i = 1, 2, 3$. Hence we have $\alpha \in Spin(8)$ and so $\alpha \in \text{Ker } p'$. Therefore $\alpha = 1$ or σ by Theorem 2.7.4. Hence we have the isomorphism

$$(E_6)_{E_1} / \{1, \sigma\} \cong SO(10).$$

Therefore the group $(E_6)_{E_1}$ is isomorphic to the group $Spin(10)$ as the universal covering group of $SO(10)$.

For $\theta \in C$, $\theta \neq 0$, we define a C -linear mapping $\phi(\theta) : \mathfrak{J}^C \rightarrow \mathfrak{J}^C$ by

$$\phi(\theta) \begin{pmatrix} \xi_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & \xi_2 & x_1 \\ x_2 & \bar{x}_1 & \xi_3 \end{pmatrix} = \begin{pmatrix} \theta^4 \xi_1 & \theta x_3 & \theta \bar{x}_2 \\ \theta \bar{x}_3 & \theta^{-2} \xi_2 & \theta^{-2} x_1 \\ \theta x_2 & \theta^{-2} \bar{x}_1 & \theta^{-2} \xi_3 \end{pmatrix}.$$

Theorem 3.10.5. *The group E_6 contains a subgroup*

$$\begin{aligned}
U(1) &= \{\alpha_{12}(t)\alpha_{13}(t) = \exp it(2E_1 - E_2 - E_3)^\sim \mid t \in \mathbf{R}\} \\
&= \{\phi(\theta) \mid \theta \in C, (\tau\theta)\theta = 1\}
\end{aligned}$$

(where $\alpha_{12}(t), \alpha_{13}(t)$ are mappings defined in Lemma 3.8.1) which is isomorphic to the usual unitary group $U(1) = \{\theta \in C \mid (\tau\theta)\theta = 1\}$.

Note that $U(1)$ is a subgroup of $(E_6)^\sigma$. From now on, we identify these two groups $U(1)$.

Lemma 3.10.6. *Two subgroups $U(1)$ and $Spin(10)$ of $(E_6)^\sigma$ are elementwise commutative.*

Proof. We consider the decomposition $\mathfrak{J}^C = \mathfrak{E}_1^C \oplus \mathfrak{J}(2, \mathfrak{C})^C \oplus (\mathfrak{J}^C)_{-\sigma}$, where

$$\begin{aligned}
\mathfrak{E}_1^C &= \{\xi E_1 \mid \xi \in C\}, \quad \mathfrak{J}(2, \mathfrak{C})^C = \{X \in \mathfrak{J}^C \mid 4E_1 \times (E_1 \times X) = X\}, \\
(\mathfrak{J}^C)_{-\sigma} &= \{X \in \mathfrak{J}^C \mid E_1 \times X = 0, \langle E_1, X \rangle = 0\}.
\end{aligned}$$

The restrictions of $\phi(\theta) \in U(1)$ to these spaces are all constant mappings:

$$\phi(\theta)|_{\mathfrak{E}_1^C} = \theta^4 1, \quad \phi(\theta)|_{\mathfrak{J}(2, \mathfrak{C})^C} = \theta^{-2} 1, \quad \phi(\theta)|_{(\mathfrak{J}^C)_{-\sigma}} = \theta 1.$$

On the other hand, $\beta \in Spin(10)$ also induces C -linear transformations of these spaces. From this, the commutativity of $\phi(\theta)$ and β : $\phi(\theta)\beta = \beta\phi(\theta)$ follows.

Theorem 3.10.7. $(E_6)^\sigma \cong (U(1) \times Spin(10))/\mathbf{Z}_4$, $\mathbf{Z}_4 = \{(1, \phi(1)), (-1, \phi(-1)), (i, \phi(-i)), (-i, \phi(i))\}$.

Proof. We define a mapping $\varphi : U(1) \times Spin(10) \rightarrow (E_6)^\sigma$ by

$$\varphi(\theta, \beta) = \phi(\theta)\beta.$$

Since $\phi(\theta) \in U(1)$ and $\beta \in Spin(10)$ are commutative (Lemma 3.10.6), we see that φ is a homomorphism. We shall show that φ is onto. For $\alpha \in (E_6)^\sigma$, there exists $\theta \in C$, $(\tau\theta)\theta = 1$ satisfying

$$\alpha E_1 = \theta^4 E_1 = \phi(\theta)E_1$$

(Lemma 3.10.1). Let $\beta = \phi(\theta)^{-1}\alpha$, then $\beta E_1 = E_1$, so $\beta \in Spin(10)$ (Theorem 3.10.4). Hence, we have $\alpha = \phi(\theta)\beta = \varphi(\theta, \beta)$ and therefore φ is onto. It is easily seen that

$$\text{Ker } \varphi = \{(\theta, \phi(\theta)^{-1}) \mid \theta \in C, \theta^4 = 1\} = \{(\theta, \phi(\theta^{-1})) \mid \theta = \pm 1, \pm i\} = \mathbf{Z}_4.$$

Thus we have the isomorphism $(U(1) \times Spin(10))/\mathbf{Z}_4 \cong (E_6)^\sigma$.

3.11. Involution γ and subgroup $(Sp(1) \times SU(6))/\mathbf{Z}_2$ of E_6

Let the C -linear mapping $\gamma : \mathfrak{J}^C \rightarrow \mathfrak{J}^C$ be the complexification of $\gamma \in G_2 \subset F_4$. Then $\gamma \in E_6$ and $\gamma^2 = 1$.

We shall study the following subgroup $(E_6)^\gamma$ of E_6 :

$$(E_6)^\gamma = \{\alpha \in E_6 \mid \gamma\alpha = \alpha\gamma\}.$$

As in Section 2.11, we use the decomposition

$$\mathfrak{J}^C = \mathfrak{J}(3, \mathbf{H})^C \oplus (\mathbf{H}^3)^C,$$

which is the complexification of the decomposition $\mathfrak{J} = \mathfrak{J}(3, \mathbf{H}) \oplus \mathbf{H}^3$. As usual we denote $\mathfrak{J}(3, \mathbf{H})$ and $\{X \in \mathfrak{J}(3, \mathbf{H}) \mid \text{tr}(X) = 0\}$ by $\mathfrak{J}_\mathbf{H}$ and $(\mathfrak{J}_\mathbf{H})_0$, respectively.

We consider the embedding $\mathbf{C} = \{x + ye_1 \mid x, y \in \mathbf{R}\} \subset \mathfrak{C}$ and, for an element $a = x + ye_1 \in \mathbf{C}$, we denote by a' the element $x + yi \in C$. Now, we define an \mathbf{R} -linear mapping $k : \mathbf{H} \rightarrow M(2, C)$ by

$$k(a + be_2) = \begin{pmatrix} a' & b' \\ -\tau b' & \tau a' \end{pmatrix}, \quad a, b \in \mathbf{C}.$$

The mapping k is naturally extended to \mathbf{R} -linear mappings

$$k : M(3, \mathbf{H}) \rightarrow M(6, C) \quad \text{and} \quad k : \mathbf{H}^3 \rightarrow M(2, 6, C).$$

We adopt the following notations.

$$J = \begin{pmatrix} J & 0 & 0 \\ 0 & J & 0 \\ 0 & 0 & J \end{pmatrix} \in M(6, C), \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and $M^* = {}^t\overline{M}$, $M \in M(3, \mathbf{H})$. Then the following four properties hold.

- (1) $k(MN) = k(M)k(N)$,
 - (2) $\tau {}^t(k(M)) = k(M^*)$,
 - (3) $J(k(M)) = (\tau(k(M)))J$,
 - (4) $k(\mathbf{a}M) = k(\mathbf{a})k(M)$,
- $M, N \in M(3, \mathbf{H}), \mathbf{a} \in \mathbf{H}^3$.

The \mathbf{R} -linear mappings $k : M(3, \mathbf{H}) \rightarrow M(6, C)$ and $k : \mathbf{H}^3 \rightarrow M(2, 6, C)$ are extended to C -linear mappings $k : M(3, \mathbf{H})^C \rightarrow M(6, C)$ and $k : (\mathbf{H}^3)^C \rightarrow M(2, 6, C)$ respectively by

$$\begin{aligned} k(M_1 + iM_2) &= k(M_1) + ik(M_2), & M_1, M_2 \in M(3, \mathbf{H}), \\ k(\mathbf{a}_1 + i\mathbf{a}_2) &= k(\mathbf{a}_1) + ik(\mathbf{a}_2), & \mathbf{a}_1, \mathbf{a}_2 \in \mathbf{H}^3. \end{aligned}$$

It is not difficult to see that they satisfy the four properties (1) \sim (4) above.

We define a C -vector space $\mathfrak{S}(6, C)$ by

$$\mathfrak{S}(6, C) = \{S \in M(6, C) \mid {}^tS = -S\}$$

and a C -linear mapping $k_J : \mathfrak{J}(3, \mathbf{H})^C \rightarrow \mathfrak{S}(6, C)$ by

$$k_J(M) = k(M)J.$$

Then k_J is well-defined. Indeed, for $M = M_1 + iM_2 \in \mathfrak{J}(3, \mathbf{H})^C$, we have

$$\begin{aligned} {}^t(k_J(M)) &= {}^t((k(M))J) = -J {}^t(k(M)) = -J {}^t(k(M_1) + ik(M_2)) \\ &= -(\tau {}^t(k(M_1)) + i\tau {}^t(k(M_2)))J = -(k(M_1^*) + ik(M_2^*))J \\ &= -(k(M_1) + ik(M_2))J = -(k(M))J = -k_J(M). \end{aligned}$$

Finally, we define Hermitian inner products $\langle S, T \rangle$ in $\mathfrak{S}(6, C)$ and $\langle P, Q \rangle$ in $M(2, 6, C)$ respectively by

$$\langle S, T \rangle = \text{tr}((\tau {}^tS)T), \quad \langle P, Q \rangle = \text{tr}((\tau {}^tP)Q).$$

Lemma 3.11.1. (1) $k : M(3, \mathbf{H})^C \rightarrow M(6, C)$, $k : (\mathbf{H}^3)^C \rightarrow M(2, 6, C)$ and $k_J : \mathfrak{J}(3, \mathbf{H})^C \rightarrow \mathfrak{S}(6, C)$ are C -linear isomorphisms.

- (2) $\langle k_J(M), k_J(N) \rangle = 2\langle M, N \rangle$, $M, N \in \mathfrak{J}(3, \mathbf{H})^C$,
- $\langle k(\mathbf{a}), k(\mathbf{b}) \rangle = 2\langle \mathbf{a}, \mathbf{b} \rangle$, $\mathbf{a}, \mathbf{b} \in (\mathbf{H}^3)^C$.
- (3) $\det(k_J(M)) = (\det M)^2$, $M \in \mathfrak{J}(3, \mathbf{H})^C$.

Proof. (1) The mapping k is injective. Indeed, if

$$k((a + be_2) + i(c + de_2)) = 0, \quad a, b, c, d \in \mathbf{C},$$

then

$$\begin{pmatrix} a' & b' \\ -\tau b' & \tau a' \end{pmatrix} + i \begin{pmatrix} c' & d' \\ -\tau d' & \tau c' \end{pmatrix} = 0.$$

From which, we have $a' = b' = c' = d' = 0$ and so $a = b = c = d = 0$. Now, k is a C -linear isomorphism, because $\dim_C(M(3, \mathbf{H})^C) = 36 = \dim_C(M(6, C))$. The other mapping can be treated analogously.

(2) For $m_k = a_k + b_k e_2$ and $n_k = c_k + d_k e_2, a_k, b_k, c_k, d_k \in \mathbf{C}, k = 1, 2$, we have, after some calculations,

$$\langle k(m_1 + in_1), k(m_2 + in_2) \rangle = 2\langle m_1 + in_1, m_2 + in_2 \rangle.$$

(3) Note that $\det(k_J(M)) = \det(k(M))$ and $\det M \in C$. Since we know that $\det S$ of a skew-symmetric matrix S is expressed in terms of the square of a polynomial with entries s_{ij} in S , we can easily see that

$$\begin{aligned} \det \begin{pmatrix} 0 & s_{12} & s_{13} & s_{14} & s_{15} & s_{16} \\ -s_{12} & 0 & s_{23} & s_{24} & s_{25} & s_{26} \\ -s_{13} & -s_{23} & 0 & s_{34} & s_{35} & s_{36} \\ -s_{14} & -s_{24} & -s_{34} & 0 & s_{45} & s_{46} \\ -s_{15} & -s_{25} & -s_{35} & -s_{45} & 0 & s_{56} \\ -s_{16} & -s_{26} & -s_{36} & -s_{46} & -s_{56} & 0 \end{pmatrix} \\ = (s_{12}s_{34}s_{56} - s_{12}s_{35}s_{46} + s_{12}s_{36}s_{45} - s_{13}s_{24}s_{56} + s_{13}s_{25}s_{46} - s_{13}s_{26}s_{45} \\ + s_{14}s_{23}s_{56} - s_{14}s_{25}s_{36} + s_{14}s_{26}s_{35} - s_{15}s_{23}s_{46} + s_{15}s_{24}s_{36} - s_{15}s_{26}s_{35} \\ + s_{16}s_{23}s_{45} - s_{16}s_{24}s_{35} + s_{16}s_{25}s_{34})^2. \end{aligned}$$

Now, since $k_J(M)$ is skew-symmetric, using the above result, we have

$$\begin{aligned} \det(k_J(M)) &= \det \begin{pmatrix} 0 & \xi_1 & -n_3 & m_3 & n_2 & \tau(m_2) \\ -\xi_1 & 0 & -\tau(m_3) & -\tau(n_3) & -m_2 & \tau(n_2) \\ n_3 & \tau(m_3) & 0 & \xi_2 & -n_1 & m_1 \\ -m_3 & \tau(n_3) & -\xi_2 & 0 & -\tau(m_1) & -\tau(n_1) \\ -n_2 & m_2 & n_1 & \tau(m_1) & 0 & \xi_3 \\ -\tau(m_2) & -\tau(n_2) & -m_1 & \tau(n_1) & -\xi_3 & 0 \end{pmatrix} \\ &= (\xi_1\xi_2\xi_3 - \xi_1n_1\tau(n_1) - \xi_1m_1\tau(m_1) - n_3\tau(n_3)\xi_3 - n_3m_2\tau(n_1) - n_3\tau(n_2)\tau(m_1) \\ &\quad - m_3\tau(m_3)\xi_3 + m_3m_2m_1 - m_3\tau(n_2)n_1 - n_2\tau(m_3)\tau(n_1) - n_2\tau(n_3)m_1 - n_2\tau(n_2)\xi_2 \\ &\quad + \tau(m_2)\tau(m_3)\tau(m_1) - \tau(m_2)\tau(n_3)n_1 - \tau(m_2)m_2\xi_2)^2. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \det \begin{pmatrix} \xi_1 & m_3 + n_3e_2 & \tau(m_2 + n_2e_2) \\ \tau(m_3 + n_3e_2) & \xi_2 & m_1 + n_1e_2 \\ m_2 + n_2e_2 & \tau(m_1 + n_1e_2) & \xi_3 \end{pmatrix} \\ = \xi_1\xi_2\xi_3 + (m_1 + n_1e_2)(m_2 + n_2e_2)(m_3 + n_3e_2) \\ + \tau((m_1 + n_1e_2)(m_2 + n_2e_2)(m_3 + n_3e_2)) - \sum_{i=1}^3 \xi_i(m_i + n_ie_2)\tau(m_i + n_ie_2) \\ = \text{the contents of in the parenthesis above.} \end{aligned}$$

We now consider a group $E_{6,\mathbf{H}}$ defined by replacing \mathfrak{C} by \mathbf{H} in the definition of the group E_6 :

$$E_{6,\mathbf{H}} = \{\alpha \in \text{Iso}_C((\mathfrak{J}_{\mathbf{H}})^C) \mid \det(\alpha M) = \det M, \langle \alpha M, \alpha N \rangle = \langle M, N \rangle\}.$$

Lemma 3.11.2. $E_{6,\mathbf{H}}$ is a connected group of dimension 35.

Proof. As in the case E_6 , the group $E_{6,\mathbf{H}}$ contains a subgroup

$$F_{4,\mathbf{H}} = \{\alpha \in E_{6,\mathbf{H}} \mid \alpha E = E\},$$

which is also defined by

$$F_{4,\mathbf{H}} = \{\alpha \in \text{Iso}_{\mathbf{R}}(\mathfrak{J}_{\mathbf{H}}) \mid \alpha(M \circ N) = \alpha M \circ \alpha N\}$$

(see Lemma 2.2.4). Moreover, it is isomorphic to the group $Sp(3)/\mathbf{Z}_2$ (Proposition 2.11.1). The Lie algebra $\mathfrak{e}_{6,\mathbf{H}}$ of the group $E_{6,\mathbf{H}}$ is given by

$$\begin{aligned} \mathfrak{e}_{6,\mathbf{H}} &= \{\phi \in \text{Hom}_C((\mathfrak{J}_{\mathbf{H}})^C) \mid (\phi M, M, M) = 0, \langle \phi M, N \rangle + \langle M, \phi N \rangle = 0\} \\ &= \{\delta + i\tilde{T} \mid \delta \in \mathfrak{f}_{4,\mathbf{H}}, T \in (\mathfrak{J}_{\mathbf{H}})_0\}, \end{aligned}$$

(see Theorem 3.2.1), so that

$$\mathfrak{e}_{6,\mathbf{H}} = \mathfrak{f}_{4,\mathbf{H}} \oplus i(\tilde{\mathfrak{J}}_{\mathbf{H}})_0.$$

Where $\mathfrak{f}_{4,\mathbf{H}} = \{\delta \in \mathfrak{e}_{6,\mathbf{H}} \mid \delta E = 0\}$ is the Lie algebra of the group $F_{4,\mathbf{H}}$ and is isomorphic to the Lie algebra $\mathfrak{sp}(3)$ by the mapping $\varphi_* : \mathfrak{sp}(3) \rightarrow \mathfrak{f}_{4,\mathbf{H}}$ given by

$$\varphi_*(C)M = CM + MC^* = [C, M], \quad M \in \mathfrak{J}_{\mathbf{H}}$$

(Proposition 2.11.1) and we see that $\dim \mathfrak{e}_{6,\mathbf{H}} = 21 + 15 = 35$. As in Theorem 3.8.3, we have a homeomorphism

$$E_{6,\mathbf{H}}/F_{4,\mathbf{H}} \simeq EIV_{\mathbf{H}} = \{X \in (\mathfrak{J}_{\mathbf{H}})^C \mid \det M = 1, \langle M, M \rangle = 3\}$$

and so we see that the group $E_{6,\mathbf{H}}$ is connected.

Proposition 3.11.3. $E_{6,\mathbf{H}} \cong SU(6)/\mathbf{Z}_2$, $\mathbf{Z}_2 = \{E, -E\}$.

Proof. Let $SU(6) = \{A \in M(6, \mathbb{C}) \mid (\tau^t A)A = E, \det A = 1\}$ and define a mapping $\varphi : SU(6) \rightarrow E_{6,\mathbf{H}}$ by

$$\varphi(A)M = k_J^{-1}(A(k_J(M))^t A), \quad M \in (\mathfrak{J}_{\mathbf{H}})^C.$$

We first have to prove that $\varphi(A) \in E_{6,\mathbf{H}}$. Indeed, we have $(\det(\varphi(A)M))^2 = \det(k_J(\varphi(A)M))$ (Lemma 3.11.1) $= \det(A(k_J(M))^t A) = \det(k_J(M)) = (\det M)^2$ (Lemma 3.11.1), and so $\det(\varphi(A)M) = \pm \det M$. On the other hand, since $E_{6,\mathbf{H}}$ is connected (Lemma 3.11.2), the sign of $\det(\varphi(A)M)$ is constant, that is, independent of A (assuming that $\det M \neq 0$). Therefore

$$\det(\varphi(A)M) = \det M.$$

Next, again we have

$$\begin{aligned} 2\langle \varphi(A)M, \varphi(A)N \rangle &= \langle k_J(\varphi(A)M), k_J(\varphi(A)N) \rangle \quad (\text{Lemma 3.11.1}) \\ &= \langle A(k_J(M))^t A, A(k_J(N))^t A \rangle = \langle k_J(M), k_J(N) \rangle \quad (\text{because } A \in SU(6)) \\ &= 2\langle M, N \rangle, \end{aligned}$$

so $\varphi(A) \in E_{6,\mathbf{H}}$. Consequently the mapping $\varphi : SU(6) \rightarrow E_{6,\mathbf{H}}$ is well-defined. Evidently φ is a homomorphism. That $\text{Ker}\varphi = \{E, -E\} = \mathbf{Z}_2$ can be easily obtained. Since the group $E_{6,\mathbf{H}}$ is connected and $\dim SU(6) = 35 = \dim E_{6,\mathbf{H}}$ (Lemma 3.11.2), φ is onto. Thus we have the isomorphism $SU(6)/\mathbf{Z}_2 \cong E_{6,\mathbf{H}}$.

Theorem 3.11.4. $(E_6)^\gamma \cong (Sp(1) \times SU(6))/\mathbf{Z}_2$, $\mathbf{Z}_2 = \{(1, E), (-1, -E)\}$.

Proof. We define a mapping $\varphi : Sp(1) \times SU(6) \rightarrow (E_6)^\gamma$ by

$$\begin{aligned} \varphi(p, A)(M + \mathbf{a}) &= k_J^{-1}(A(k_J(M))^t A) + p\mathbf{a}k^{-1}(\tau^t A), \\ M + \mathbf{a} &\in (\mathfrak{J}_H)^C \oplus (\mathbf{H}^3)^C = \mathfrak{J}^C. \end{aligned}$$

We first need to prove that $\varphi(p, A) \in (E_6)^\gamma$.

Claim 1. ${}^t\varphi(p, A)^{-1} = \tau\varphi(p, A)\tau$.

Proof. we have

$$\begin{aligned} 2\langle \tau^t \varphi(p, A)\tau(M + \mathbf{a}), N + \mathbf{b} \rangle & \quad M + \mathbf{a}, N + \mathbf{b} \in \mathfrak{J}^C \\ &= 2\langle M + \mathbf{a}, \varphi(p, A)(N + \mathbf{b}) \rangle \\ &= 2\langle M + \mathbf{a}, k_J^{-1}(A(k_J(N))^t A) + p\mathbf{b}k^{-1}(\tau^t A) \rangle \\ &= 2\langle M, k_J^{-1}(A(k_J(N))^t A) \rangle + 4\langle \mathbf{a}, p\mathbf{b}k^{-1}(\tau^t A) \rangle \\ &= \langle k_J M, A(k_J(N))^t A \rangle + 2\langle k(\overline{p}\mathbf{a}), (k\mathbf{b})\tau^t A \rangle \\ &= \langle \tau^t A(k_J(M))\tau A, k_J(N) \rangle + 2\langle k(\overline{p}\mathbf{a})A, k\mathbf{b} \rangle \\ &= 2\langle k_J^{-1}(\tau^t A(k_J(M))\tau A), N \rangle + 4\langle \overline{p}\mathbf{a}k^{-1}(A), \mathbf{b} \rangle \\ &= 2\langle k_J^{-1}(\tau^t A(k_J(M))\tau A + \overline{p}\mathbf{a}k^{-1}(A)), N + \mathbf{b} \rangle \\ &= 2\langle \varphi(\overline{p}, \tau^t A)(M + \mathbf{a}), N + \mathbf{b} \rangle, \end{aligned}$$

and so we have $\tau^t \varphi(p, A)\tau = \varphi(\overline{p}, \tau^t A)$, which implies that ${}^t\varphi(p, A)^{-1} = \tau\varphi(p, A)\tau$.

Claim 2. $\varphi(p, A) \in (E_6)^\gamma$.

Proof. We will first show that $\alpha = \varphi(p, A)$ satisfies

$$\alpha X \times \alpha Y = {}^t\alpha^{-1}(X \times Y), \quad X, Y \in \mathfrak{J}^C.$$

Recall the equality

$$(M + \mathbf{a}) \times (N + \mathbf{b}) = \left(M \times N - \frac{1}{2}(\mathbf{a}^* \mathbf{b} + \mathbf{b}^* \mathbf{a}) \right) - \frac{1}{2}(\mathbf{a}N + \mathbf{b}M).$$

The equality $\alpha M \times \alpha N = {}^t\alpha^{-1}(M \times N)$ is evident from $\det(\alpha M) = \det M$ (Proposition 3.11.3). Now, using the equalities

$$k_J^{-1}(M) = -k^{-1}(MJ) = -k^{-1}(J(\tau M)), \quad \tau k^{-1}(M) = -k^{-1}(JMJ),$$

we have

$$\begin{aligned}
(\alpha \mathbf{a})^*(\alpha \mathbf{b}) &= (p \mathbf{a} k^{-1}(\tau^t A))^*(p \mathbf{b} k^{-1}(\tau^t A)) \\
&= k^{-1}(A) \mathbf{a}^* \mathbf{b} k^{-1}(\tau^t A), \\
\tau \alpha \tau(\mathbf{a}^* \mathbf{b}) &= \tau(k_J^{-1}(A(k_J \tau(\mathbf{a}^* \mathbf{b}))^t A)) \\
&= -\tau k^{-1}(J \tau(A(k(\tau(\mathbf{a}^* \mathbf{b})))J^t A)) \\
&= -\tau k^{-1}(J \tau(A J \tau k(\mathbf{a}^* \mathbf{b})^t A)) \\
&= -\tau k^{-1}(J \tau A J k(\mathbf{a}^* \mathbf{b}) \tau^t A) \\
&= -k^{-1}(A \tau k(\mathbf{a}^* \mathbf{b}) J^t A J) = k^{-1}(A) \mathbf{a}^* \mathbf{b} k^{-1}(\tau^t A), \\
(\alpha \mathbf{a})(\alpha N) &= (p \mathbf{a} k^{-1}(\tau^t A)) k_J^{-1}(A(k_J(N))^t A) \\
&= -p \mathbf{a} k^{-1}(\tau^t A) k^{-1}(A(k(N))J^t A J) \\
&= -p \mathbf{a} N k^{-1}(J^t A J) = p \mathbf{a} N \tau k^{-1}(\tau^t A), \\
\tau \alpha \tau(\mathbf{a} N) &= \tau(p \tau(\mathbf{a} N) k^{-1}(\tau^t A)) = p \mathbf{a} N \tau k^{-1}(\tau^t A).
\end{aligned}$$

We have therefore shown that $\varphi(p, A) \in E_6$. Clearly, $\gamma \varphi(p, A) = \varphi(p, A) \gamma$, so that $\varphi(p, A) \in (E_6)^\gamma$.

We will return to the proof of Theorem 3.11.4. Evidently φ is a homomorphism. We shall now show that φ is onto. Let $\alpha \in (E_6)^\gamma$. Since the restriction $\alpha' = \alpha|(\mathfrak{J}_H)^C$ of α to $(\mathfrak{J}_H)^C$ belongs to $E_{6,H}$, there exists $A \in SU(6)$ such that $\alpha' = \varphi(A)$ (Proposition 3.11.3). If we put $\beta = \varphi(1, A)^{-1} \alpha$, then $\beta|(\mathfrak{J}_H)^C = 1$. Hence, by the same argument as in Theorem 2.11.2, there exists $p \in Sp(1)$ such that $\beta = \varphi(p, E)$, and we obtain

$$\alpha = \varphi(1, A) \beta = \varphi(1, A) \varphi(p, E) = \varphi(p, A).$$

Therefore φ is onto. $\text{Ker } \varphi = \{(1, E), (-1, -E)\} = \mathbf{Z}_2$ can be easily obtained. Thus we have the isomorphism $(Sp(1) \times SU(6))/\mathbf{Z}_2 \cong (E_6)^\gamma$.

3.12. Involution $\tau\gamma$ and subgroup $Sp(4)/\mathbf{Z}_2$ of E_6

We consider an involutive complex conjugate linear transformation $\tau\gamma$ of \mathfrak{J}^C , and we shall study the following subgroup $(E_6)^{\tau\gamma}$ of E_6 :

$$\begin{aligned}
(E_6)^{\tau\gamma} &= \{\alpha \in E_6 \mid \tau\gamma\alpha = \alpha\tau\gamma\} \\
&= \{\alpha \in E_6 \mid \gamma\lambda(\alpha)\gamma = \alpha\} = (E_6)^{\lambda\gamma}.
\end{aligned}$$

For this end, we consider \mathbf{R} -vector subspaces $(\mathfrak{J}^C)_{\tau\gamma}$ and $(\mathfrak{J}^C)_{-\tau\gamma}$ of \mathfrak{J}^C , which are eigenspaces of $\tau\gamma$, respectively by

$$\begin{aligned}
(\mathfrak{J}^C)_{\tau\gamma} &= \{X \in \mathfrak{J}^C \mid \tau\gamma X = X\} \\
&= \left\{ \begin{pmatrix} \mu_1 & m_3 & \overline{m}_2 \\ \overline{m}_3 & \mu_2 & m_1 \\ m_2 & \overline{m}_1 & \mu_3 \end{pmatrix} + i \begin{pmatrix} 0 & a_3 e_4 & -a_2 e_4 \\ -a_3 e_4 & 0 & a_1 e_4 \\ a_2 e_4 & -a_1 e_4 & 0 \end{pmatrix} \mid \begin{array}{l} \mu_i \in \mathbf{R} \\ m_i, a_i \in \mathbf{H} \end{array} \right\}
\end{aligned}$$

$$\begin{aligned}
&= \{M + iF(\mathbf{a}e_4) \mid M \in \mathfrak{J}(3, \mathbf{H}), \mathbf{a} \in \mathbf{H}^3\} \\
&= \mathfrak{J}_{\mathbf{H}} \oplus i\mathbf{H}^3, \\
(\mathfrak{J}^C)_{-\tau\gamma} &= \{X \in \mathfrak{J}^C \mid \tau\gamma X = -X\} \\
&= \left\{ i \begin{pmatrix} \mu_1 & m_3 & \overline{m}_2 \\ \overline{m}_3 & \mu_2 & m_1 \\ m_2 & \overline{m}_1 & \mu_3 \end{pmatrix} + \begin{pmatrix} 0 & a_3e_4 & -a_2e_4 \\ -a_3e_4 & 0 & a_1e_4 \\ a_2e_4 & -a_1e_4 & 0 \end{pmatrix} \mid \begin{array}{l} \mu_i \in \mathbf{R} \\ m_i, a_i \in \mathbf{H} \end{array} \right\} \\
&= \{iM + F(\mathbf{a}e_4) \mid M \in \mathfrak{J}(3, \mathbf{H}), \mathbf{a} \in \mathbf{H}^3\} \\
&= i\mathfrak{J}_{\mathbf{H}} \oplus \mathbf{H}^3,
\end{aligned}$$

The spaces $(\mathfrak{J}^C)_{\tau\gamma}$ and $(\mathfrak{J}^C)_{-\tau\gamma}$ are invariant under the action of $(E_6)^{\tau\gamma}$ and we have the decomposition of \mathfrak{J}^C :

$$\mathfrak{J}^C = (\mathfrak{J}^C)_{\tau\gamma} \oplus (\mathfrak{J}^C)_{-\tau\gamma} = (\mathfrak{J}^C)_{\tau\gamma} \oplus i(\mathfrak{J}^C)_{\tau\gamma}.$$

In particular, \mathfrak{J}^C is the complexification of $(\mathfrak{J}^C)_{\tau\gamma}$: $\mathfrak{J}^C = ((\mathfrak{J}^C)_{\tau\gamma})^C$.

In the \mathbf{R} -vector space

$$\mathfrak{J}(4, \mathbf{H}) = \{P \in M(4, \mathbf{H}) \mid P^* = P\},$$

we define the Jordan multiplication $P \circ Q$ and an inner product (P, Q) respectively by

$$P \circ Q = \frac{1}{2}(PQ + QP), \quad (P, Q) = \text{tr}(P \circ Q).$$

The group $Sp(4)$ acts on $\mathfrak{J}(4, \mathbf{H})$ by the mapping $\mu : Sp(4) \times \mathfrak{J}(4, \mathbf{H}) \rightarrow \mathfrak{J}(4, \mathbf{H})$, $\mu(A, P) = APA^*$. Then we have

$$\begin{aligned}
A(P \circ Q)A^* &= APA^* \circ AQA^*, \\
(APA^*, AQA^*) &= (P, Q),
\end{aligned} \quad A \in Sp(4), P, Q \in \mathfrak{J}(4, \mathbf{H}).$$

The quaternion projective space \mathbf{HP}_3 is defined by

$$\begin{aligned}
\mathbf{HP}_3 &= \{P \in \mathfrak{J}(4, \mathbf{H}) \mid P^2 = P, \text{tr}(P) = 1\} \\
&= \left\{ AE_1A^* \mid A \in Sp(4), E_1 = \text{diag}(1, 0, 0, 0) \in M(4, \mathbf{H}) \right\}.
\end{aligned}$$

Finally, in the complexification $\mathfrak{J}(4, \mathbf{H})^C$ of $\mathfrak{J}(4, \mathbf{H})$, we extend naturally the Jordan multiplication $P \circ Q$ and the inner product (P, Q) and further define a Hermitian inner product $\langle P, Q \rangle$ by

$$\langle P, Q \rangle = (\tau P, Q).$$

The action of $Sp(4)$ on $\mathfrak{J}(4, \mathbf{H})$ is also naturally extended to $\mathfrak{J}(4, \mathbf{H})^C$:

$$A(X_1 + iX_2)A^* = AX_1A^* + iAX_2A^*, \quad A \in Sp(4), X_1, X_2 \in \mathfrak{J}(4, \mathbf{H}).$$

Then we have

$$\langle APA^*, AQA^* \rangle = \langle P, Q \rangle, \quad P, Q \in \mathfrak{J}(4, \mathbf{H})^C.$$

We denote by $\mathfrak{J}(4, \mathbf{H})_0$ the space $\{P \in \mathfrak{J}(4, \mathbf{H}) \mid \text{tr}(P) = 0\}$ and by $\mathfrak{J}(4, \mathbf{H})_0^C$ its complexification.

Definition. We define a C -linear mapping $g : \mathfrak{J}^C \rightarrow \mathfrak{J}(4, \mathbf{H})_0^C$ by

$$g(M + \mathbf{a}) = \begin{pmatrix} \frac{1}{2}\text{tr}(M) & i\mathbf{a} \\ i\mathbf{a}^* & M - \frac{1}{2}\text{tr}(M)E \end{pmatrix}, \quad M + \mathbf{a} \in (\mathfrak{J}_{\mathbf{H}})^C \oplus (\mathbf{H}^3)^C = \mathfrak{J}^C.$$

The restriction of g to $(\mathfrak{J}^C)_{\tau\gamma}$ is given by

$$g(M + i\mathbf{a}) = \begin{pmatrix} \text{tr}(M) & -\mathbf{a} \\ -\mathbf{a}^* & M \end{pmatrix} - \frac{1}{2}\text{tr}(M)E, \quad M + i\mathbf{a} \in \mathfrak{J}_{\mathbf{H}} \oplus i\mathbf{H}^3 = (\mathfrak{J}^C)_{\tau\gamma}.$$

Note that the mapping g in the above definition is the complexification of this restriction.

Lemma 3.12.1. *The mapping $g : \mathfrak{J}^C \rightarrow \mathfrak{J}(4, \mathbf{H})_0^C$ is a C -linear isomorphism and satisfies*

$$\begin{aligned} gX \circ gY &= g(\gamma(X \times Y)) + \frac{1}{4}(\gamma X, Y)E, \\ (gX, gY) &= (\gamma X, Y), \end{aligned} \quad X, Y \in \mathfrak{J}^C.$$

Moreover, g is an isometry with respect to the inner product $\langle X, Y \rangle$:

$$\langle gX, gY \rangle = \langle X, Y \rangle, \quad X, Y \in \mathfrak{J}^C.$$

The restriction of g to $(\mathfrak{J}^C)_{\tau\gamma}$ induces an \mathbf{R} -linear isomorphism $g : (\mathfrak{J}^C)_{\tau\gamma} \rightarrow \mathfrak{J}(4, \mathbf{H})_0$.

Proof. It is not difficult to see that g is well-defined and that it is injective. Since $\dim_C \mathfrak{J}^C = 27 = \dim_C \mathfrak{J}(4, \mathbf{H})_0^C$, g is a C -linear isomorphism. Now, for $X = M + \mathbf{a}$, $Y = N + \mathbf{b} \in (\mathfrak{J}_{\mathbf{H}})^C \oplus (\mathbf{H}^3)^C = \mathfrak{J}^C$, we have

$$\begin{aligned} g(\gamma(X \times Y)) &= g(\gamma((M + \mathbf{a}) \times (N + \mathbf{b}))) \\ &= g((M - \mathbf{a}) \times (N - \mathbf{b})) \\ &= g((M \times N - \frac{1}{2}(\mathbf{a}^* \mathbf{b} + \mathbf{b}^* \mathbf{a})) + \frac{1}{2}(\mathbf{a}N + \mathbf{b}M)) \\ &= \begin{pmatrix} \frac{1}{2}\text{tr}(M \times N) - \frac{1}{2}(\mathbf{a}, \mathbf{b}) & \frac{i}{2}(\mathbf{a}N + \mathbf{b}M) \\ \frac{i}{2}(\mathbf{a}N + \mathbf{b}M)^* & M \times N - \frac{1}{2}(\mathbf{a}^* \mathbf{b} + \mathbf{b}^* \mathbf{a}) - \frac{1}{2}(\text{tr}(M \times N) - (\mathbf{a}, \mathbf{b}))E \end{pmatrix} \\ &= \dots \\ &= g(M + \mathbf{a}) \circ g(N + \mathbf{b}) - \left(\frac{1}{4}(M, N) - \frac{1}{2}(\mathbf{a}, \mathbf{b}) \right) E \\ &= g(M + \mathbf{a}) \circ g(N + \mathbf{b}) - \frac{1}{4}(\gamma(M + \mathbf{a}), (N + \mathbf{b}))E \\ &= gX \circ gY - \frac{1}{4}(\gamma X, Y)E. \end{aligned}$$

Thus the first equality $gX \circ gY = g(\gamma(X \times Y)) + \frac{1}{4}(\gamma X, Y)E$ is proved. Taking the traces of the both sides, we have

$$(gX, gY) = (\gamma X, Y), \quad X, Y \in \mathfrak{J}^C. \quad (\text{i})$$

It is easily seen that the restriction $g : (\mathfrak{J}^C)_{\tau\gamma} \rightarrow \mathfrak{J}(4, \mathbf{H})_0$ of g is an \mathbf{R} -isomorphism. Finally, if we note that $(\gamma X, Y) = \langle X, Y \rangle$ for $X, Y \in (\mathfrak{J}^C)_{\tau\gamma}$, then we can easily show that

$$\langle g(X_1 + iX_2), g(Y_1 + iY_2) \rangle = \langle X_1 + iX_2, Y_1 + iY_2 \rangle, \quad X_i, Y_i \in (\mathfrak{J}^C)_{\tau\gamma}$$

from (i). Thus Lemma 3.12.1 is proved.

Theorem 3.12.2. $(E_6)^{\tau\gamma} \cong Sp(4)/\mathbf{Z}_2$, $\mathbf{Z}_2 = \{E, -E\}$.

Proof. We define a mapping $\varphi : Sp(4) \rightarrow (E_6)^{\tau\gamma}$ by

$$\varphi(A)X = g^{-1}(A(gX)A^*), \quad X \in \mathfrak{J}^C.$$

We first have to prove that $\varphi(A) \in (E_6)^{\tau\gamma}$. Let $Z = \varphi(A)X$ and use Lemma 3.12.1, then we have

$$\begin{aligned} 3\det(\varphi(A)X) &= 3\det Z = (Z \times Z, Z) = (g(\gamma(Z \times Z)), gZ) \\ &= \left(gZ \circ gZ - \frac{1}{4}(\gamma Z, Z)E, gZ \right) \\ &= \left(gZ \circ gZ - \frac{1}{4}(gZ, gZ)E, gZ \right) \\ &= \left(A(gX)A^* \circ A(gX)A^* - \frac{1}{4}(A(gX)A^*, A(gX)A^*)E, A(gX)A^* \right) \\ &= (gX \circ gX - \frac{1}{4}(gX, gX)E, gX) \\ &= (gX \circ gX - \frac{1}{4}(\gamma X, X)E, gX) \\ &= (g(\gamma(X \times X)), gX) = (X \times X, X) = 3\det X, \end{aligned}$$

and

$$\begin{aligned} \langle \varphi(A)X, \varphi(A)Y \rangle &= \langle g\varphi(A)X, g\varphi(A)Y \rangle \\ &= \langle A(gX)A^*, A(gY)A^* \rangle = \langle gX, gY \rangle = \langle X, Y \rangle. \end{aligned}$$

Hence $\varphi(A) \in E_6$. To prove $\varphi(A) \in (E_6)^{\tau\gamma}$, namely, $\tau\gamma\varphi(A) = \varphi(A)\tau\gamma$, it is sufficient to show that

$$\tau\gamma\varphi(A)\tau\gamma X = \varphi(A)X, \quad X \in (\mathfrak{J}^C)_{\tau\gamma},$$

since $\mathfrak{J}^C = ((\mathfrak{J}^C)_{\tau\gamma})^C$. However this is evident. Indeed, if $X \in (\mathfrak{J}^C)_{\tau\gamma}$, then $gX \in \mathfrak{J}(4, \mathbf{H})_0$, so that $\varphi(A)X \in (\mathfrak{J}^C)_{\tau\gamma}$. Hence $\tau\gamma\varphi(A)\tau\gamma X = \tau\gamma\varphi(A)X = \varphi(A)X$. Evidently φ is a homomorphism. We shall show that φ is onto. For $\alpha \in (E_6)^{\tau\gamma}$, we have

$$(g(\alpha E))^2 = g(\alpha E) + \frac{3}{4}E.$$

Certainly,

$$\begin{aligned}
(g(\alpha E))^2 &= g(\alpha E) \circ g(\alpha E) = g(\gamma(\alpha E \times \alpha E)) + \frac{1}{4}(\gamma \alpha E, \alpha E)E \text{ (Lemma 3.12.1)} \\
&= g(\gamma \tau \alpha \tau E) + \frac{1}{4}\langle \tau \gamma \alpha E, \alpha E \rangle E = g(\alpha \tau \gamma E) + \frac{1}{4}\langle \alpha \tau \gamma E, \alpha E \rangle E \\
&= g(\alpha E) + \frac{1}{4}\langle \alpha E, \alpha E \rangle E = g(\alpha E) + \frac{1}{4}\langle E, E \rangle E = g(\alpha E) + \frac{3}{4}E.
\end{aligned}$$

Now, let

$$P = \frac{1}{4}(2g(\alpha E) + E),$$

then we have $P \in \mathfrak{J}(4, \mathbf{H})$ and $P^2 = P$, $\text{tr}(P) = 1$, that is, $P \in \mathbf{HP}^3$. Indeed,

$$\begin{aligned}
P^2 &= \frac{1}{16}(4(g(\alpha E))^2 + 4g(\alpha E) + E) = \frac{1}{4}(2g(\alpha E) + E) = P, \\
\text{tr}(P) &= \frac{1}{4}\text{tr}(2g(\alpha E)) + \frac{1}{4}\text{tr}(E) = 0 + 1 = 1.
\end{aligned}$$

Hence there exists $A \in Sp(4)$ such that

$$P = AE_1A^*.$$

Since $gE = 2E_1 - \frac{1}{2}E$, we have

$$\begin{aligned}
\varphi(A)E &= g^{-1}(A(gE)A^*) = g^{-1}\left(A\left(2E_1 - \frac{1}{2}E\right)A^*\right) = g^{-1}\left(2AE_1A^* - \frac{1}{2}E\right) \\
&= g^{-1}(g(\alpha E)) = \alpha E.
\end{aligned}$$

Putting $\beta = \varphi(A)^{-1}\alpha$, we have $\beta E = E$, and so $\beta \in F_4$ (Theorem 3.7.1). Also β satisfies $\tau\gamma\beta = \beta\tau\gamma$ and $\tau\beta = \beta\tau$ (Theorem 3.7.1), hence $\gamma\beta = \beta\gamma$, and therefore $\beta \in (F_4)^\gamma$. From Theorem 2.11.2, there exist $p \in Sp(1)$ and $D \in Sp(3)$ such that

$$\beta(M + \mathbf{a}) = DMD^* + p\mathbf{a}D^*, \quad M + \mathbf{a} \in \mathfrak{J}_{\mathbf{H}} \oplus \mathbf{H}^3 = \mathfrak{J}.$$

Let $B = \begin{pmatrix} p & 0 \\ 0 & D \end{pmatrix}$, then $B \in Sp(4)$ and we have

$$\beta = \varphi(B).$$

Certainly, for $M + \mathbf{a} \in (\mathfrak{J}_{\mathbf{H}})^C \oplus (\mathbf{H}^3)^C = \mathfrak{J}^C$,

$$\begin{aligned}
\varphi(B)(M + \mathbf{a}) &= g^{-1}(Bg(M + \mathbf{a})B^*) \\
&= g^{-1}\left(\begin{pmatrix} p & 0 \\ 0 & D \end{pmatrix} \left(\begin{pmatrix} \frac{1}{2}\text{tr}(M) & i\mathbf{a} \\ i\mathbf{a}^* & M - \frac{1}{2}\text{tr}(M)E \end{pmatrix} \begin{pmatrix} \bar{p} & 0 \\ 0 & D^* \end{pmatrix} \right)\right) \\
&= g^{-1}\left(\begin{pmatrix} \frac{1}{2}\text{tr}(M) & ip\mathbf{a}D^* \\ iD\mathbf{a}^*\bar{p} & DMD^* - \frac{1}{2}\text{tr}(M)E \end{pmatrix}\right) \\
&= DMD^* + p\mathbf{a}D^* = \beta(M + \mathbf{a}).
\end{aligned}$$

Hence we have

$$\alpha = \varphi(A)\beta = \varphi(A)\varphi(B) = \varphi(AB), \quad AB \in Sp(4),$$

so that φ is onto. $\text{Ker}\varphi = \{E, -E\} = \mathbf{Z}_2$ can be easily obtained. Thus we have the isomorphism $Sp(4)/\mathbf{Z}_2 \cong (E_6)^{\tau\gamma}$.

3.13. Automorphism w of order 3 and subgroup $(SU(3) \times SU(3) \times SU(3))/\mathbf{Z}_3$ of E_6

Let the C -linear mapping $w : \mathfrak{J}^C \rightarrow \mathfrak{J}^C$ be the complexification of $w \in G_2 \subset F_4$ of Section 2.12. Then $w \in E_6$ and $w^3 = 1$.

We shall study the following subgroup $(E_6)^w$ of E_6 :

$$(E_6)^w = \{\alpha \in E_6 \mid w\alpha = \alpha w\}.$$

As in Section 2.12, we identify

$$\mathfrak{J}(3, \mathbf{C})^C \oplus M(3, \mathbf{C})^C = \mathfrak{J}^C.$$

For convenience, we denote $\mathfrak{J}(3, \mathbf{C})$ and $\{X \in \mathfrak{J}(3, \mathbf{C}) \mid \text{tr}(X) = 0\}$ by $\mathfrak{J}_\mathbf{C}$ and $(\mathfrak{J}_\mathbf{C})_0$, respectively.

The group $E_{6,\mathbf{C}}$ is defined to be obtained by replacing \mathfrak{J}^C with $(\mathfrak{J}_\mathbf{C})^C$ in the definition of the group E_6 :

$$E_{6,\mathbf{C}} = \{\alpha \in \text{Iso}_C((\mathfrak{J}_\mathbf{C})^C) \mid \det(\alpha X) = \det X, \langle \alpha X, \alpha Y \rangle = \langle X, Y \rangle\}.$$

As in the case E_6 , the group $E_{6,\mathbf{C}}$ contains a subgroup

$$F_{4,\mathbf{C}} = \{\alpha \in E_{6,\mathbf{C}} \mid \alpha E = E\},$$

which is also defined by the group $F_{4,\mathbf{C}} = \{\alpha \in \text{Iso}_\mathbf{R}(\mathfrak{J}_\mathbf{C}) \mid \alpha(X \circ Y) = \alpha X \circ \alpha Y\}$, moreover, it is isomorphic to the group $(SU(3)/\mathbf{Z}_3) \cdot \mathbf{Z}_2$ (Proposition 2.12.1). The Lie algebra $\mathfrak{e}_{6,\mathbf{C}}$ of the group $E_{6,\mathbf{C}}$ is

$$\begin{aligned} \mathfrak{e}_{6,\mathbf{C}} &= \{\phi \in \text{Hom}_C((\mathfrak{J}_\mathbf{C})^C) \mid (\phi X, X, X) = 0, \langle \phi X, Y \rangle + \langle X, \phi Y \rangle = 0\} \\ &= \{\delta + i\tilde{T} \mid \delta \in \mathfrak{f}_{4,\mathbf{C}}, T \in (\mathfrak{J}_\mathbf{C})_0\} \end{aligned}$$

(Theorem 3.2.4), where $\mathfrak{f}_{4,\mathbf{C}} = \{\delta \in \mathfrak{e}_{6,\mathbf{C}} \mid \delta E = 0\}$ is the Lie algebra of the group $F_{4,\mathbf{C}}$. In particular, the dimension of $\mathfrak{e}_{6,\mathbf{C}}$ is

$$\dim(\mathfrak{e}_{6,\mathbf{C}}) = 8 + 8 = 16.$$

As in Theorem 3.8.3, we see that the space

$$EIV_\mathbf{C} = \{X \in (\mathfrak{J}_\mathbf{C})^C \mid \det X = 1, \langle X, X \rangle = 3\}$$

is connected and we have the homeomorphism

$$E_{6,\mathbf{C}}/F_{4,\mathbf{C}} \simeq EIV_{\mathbf{C}}.$$

Lemma 3.13.1. $E_{6,\mathbf{C}}$ has at most two connected components (in reality has two connected components (Proposition 3.13.4)).

Proof. From the exact sequence $\pi_0(F_{4,\mathbf{C}}) \rightarrow \pi_0(E_{6,\mathbf{C}}) \rightarrow \pi_0(EIV_{\mathbf{C}})$, that is, $\mathbf{Z}_2 \rightarrow \pi_0(E_{6,\mathbf{C}}) \rightarrow 0$ (Proposition 2.12.1), we see that $\pi_0(E_{6,\mathbf{C}})$ is 0 or \mathbf{Z}_2 .

We define mappings $h : \mathbf{C} \oplus \mathbf{C} \rightarrow \mathbf{C}^C$ and $h : M(3, \mathbf{C}) \oplus M(3, \mathbf{C}) \rightarrow M(3, \mathbf{C})^C$ respectively by

$$\begin{aligned} h(a, b) &= \frac{a+b}{2} + i\frac{a-b}{2}e_1 = \iota a + \bar{\iota}b, \\ h(A, B) &= \frac{A+B}{2} + i\frac{A-B}{2}e_1 = \iota A + \bar{\iota}B, \end{aligned} \quad \iota = \frac{1+ie_1}{2}.$$

Lemma 3.13.2. The mappings $h : \mathbf{C} \oplus \mathbf{C} \rightarrow \mathbf{C}^C$ and $h : M(3, \mathbf{C}) \oplus M(3, \mathbf{C}) \rightarrow M(3, \mathbf{C})^C$ satisfy the following four conditions.

(1) Both are \mathbf{C} -linear isomorphisms, that is, they are injective and satisfy

$$h(a, b) + h(a', b') = h(a + a', b + b'), \quad h(ca, cb) = ch(a, b), \quad c \in \mathbf{C},$$

$$h(A, B) + h(A', B') = h(A + A', B + B'), \quad h(cA, cB) = h(A, B), \quad c \in \mathbf{C}.$$

(2) $h(a, b)h(a', b') = h(aa', bb')$, $h(A, B)h(A', B') = h(AA', BB')$.

(3) $\tau h(a, b) = h(b, a)$, $\overline{h(a, b)} = h(\bar{b}, \bar{a})$.

$$\tau h(A, B) = h(B, A), \quad \overline{h(A, B)} = h(\bar{B}, \bar{A}), \quad h(A, B)^* = h(B^*, A^*).$$

(4) $\det(h(A, B)) = h(\det A, \det B)$.

Proof. It is easy to prove, noting that $\iota^2 = \iota, \bar{\iota}^2 = \bar{\iota}, \iota + \bar{\iota} = 1$.

Lemma 3.13.3. $\mathfrak{e}_{6,\mathbf{C}} \cong \mathfrak{su}(3) \oplus \mathfrak{su}(3)$.

Proof. The mapping $\phi_{\mathbf{C}} : \mathfrak{su}(3) \oplus \mathfrak{su}(3) \rightarrow \mathfrak{e}_{6,\mathbf{C}}$,

$$\phi_{\mathbf{C}}(C, D)X = h(C, D)X + Xh(C, D)^*, \quad X \in (\mathfrak{J}_{\mathbf{C}})^C$$

gives an isomorphism as Lie algebras. This is the direct consequence of the following Proposition 3.13.4, so we will omit its proof here.

We define an action of the group $\mathbf{Z}_2 = \{1, \epsilon\}$ on $SU(3) \times SU(3)$ by

$$\epsilon(A, B) = (\bar{B}, \bar{A}),$$

and let $(SU(3) \times SU(3)) \cdot \mathbf{Z}_2$ be the semi-direct product of the groups $SU(3) \times SU(3)$ and \mathbf{Z}_2 under this action.

Proposition 3.13.4. $E_{6,C} \cong ((SU(3) \times SU(3))/\mathbf{Z}_3) \cdot \mathbf{Z}_2$, $\mathbf{Z}_3 = \{(E, E), (\omega_1 E, \omega_1 E), (\omega_1^2 E, \omega_1^2 E)\}$, $\omega_1 = -\frac{1}{2} + \frac{\sqrt{3}}{2}e_1$.

Proof. We define a mapping $\varphi : (SU(3) \times SU(3)) \cdot \mathbf{Z}_2 \rightarrow E_{6,C}$ by

$$\begin{aligned}\varphi((A, B), 1)X &= h(A, B)Xh(A, B)^*, \\ \varphi((A, B), \epsilon)X &= h(A, B)\overline{X}h(A, B)^*,\end{aligned}\quad X \in (\mathfrak{J}_C)^C.$$

First we have to show that $\alpha = \varphi((A, B), 1) \in E_{6,C}$. Using $\det(h(A, B)) = h(\det A, \det B)$ (Lemma 3.13.2.(4)) $= h(1, 1) = 1$ and $\tau h(A, B)^* h(A, B) = h(A^*, B^*) h(A, B)$ (Lemma 3.13.2.(3)) $= h(A^* A, B^* B) = h(E, E) = E$, we have

$$\begin{aligned}\det(\alpha X) &= (\det(h(A, B)))(\det X)(\det(h(A, B)^*)) = \det X, \\ \langle \alpha X, \alpha Y \rangle &= \langle h(A, B)Xh(A, B)^*, h(A, B)Yh(A, B)^* \rangle \\ &= (\tau h(A, B)\tau X\tau h(A, B)^*, h(A, B)Yh(A, B)^*) \\ &= (\tau X, Y) = \langle X, Y \rangle.\end{aligned}$$

Hence $\alpha \in E_{6,C}$. Since $\varphi((E, E), \epsilon) = \epsilon \in G_{2,C} (= \text{Aut}(\mathbf{C})) \subset F_{4,C} \subset E_{6,C}$, we also have $\varphi((A, B), \epsilon) = \varphi((A, B), 1)\varphi((E, E), \epsilon) \in E_{6,C}$. Next, we shall show that φ is a homomorphism. Indeed, for instance,

$$\begin{aligned}\varphi((A, B), \epsilon)\varphi((C, D), 1)X &= \varphi((A, B), \epsilon)(h(C, D)Xh(C, D)^*) \\ &= h(A, B)\overline{h(C, D)Xh(C, D)^*}h(A, B)^* \\ &= h(A, B)h(\overline{D}, \overline{C})\overline{X}h(\overline{D}, \overline{C})^*h(A, B)^* \quad (\text{Lemma 3.13.3.(3)}) \\ &= h(A\overline{D}, B\overline{C})\overline{X}h(A\overline{D}, B\overline{C})^* = \varphi((A\overline{D}, B\overline{C}), \epsilon)X \\ &= \varphi((A, B)\epsilon(C, D), \epsilon)X, \quad X \in (\mathfrak{J}_C)^C.\end{aligned}$$

That $\text{Ker } \varphi = \{(E, E), (\omega_1 E, \omega_1 E), (\omega_1^2 E, \omega_1^2 E)\} \times 1 = \mathbf{Z}_3 \times 1$ can be easily obtained. In particular, $\text{Ker } \varphi$ is discrete. Hence φ induces an injective homomorphism

$$\varphi_* : \mathfrak{su}(3) \oplus \mathfrak{su}(3) \rightarrow \mathfrak{e}_{6,C}.$$

In particular, $\dim(\mathfrak{su}(3) \oplus \mathfrak{su}(3)) = \dim(\mathfrak{e}_{6,C})$, so φ_* is an isomorphism (φ_* coincides with ϕ_C of Lemma 3.13.3). Hence, φ induces the surjection $\varphi : SU(3) \times SU(3) \rightarrow (E_{6,C})_0$ (which denotes the connected component of $E_{6,C}$ containing the identity 1). However $\epsilon = \varphi((E, E), \epsilon) \notin (E_{6,C})_0$. Certainly, for any $A, B \in SU(3)$,

$$h(A, B)Xh(A, B)^* = \overline{X}, \quad X \in (\mathfrak{J}_C)^C$$

does not hold. Therefore $E_{6,C}$ has just two connected components (Lemma 3.13.1). Consequently, $\varphi : (SU(3) \times SU(3)) \cdot \mathbf{Z}_2 \rightarrow E_{6,C}$ is onto and we have the isomorphism $((SU(3) \times SU(3))/\mathbf{Z}_3) \cdot \mathbf{Z}_2 \cong E_{6,C}$.

Theorem 3.13.5. $(E_6)^w \cong (SU(3) \times SU(3) \times SU(3))/\mathbf{Z}_3$, $\mathbf{Z}_3 = \{(E, E, E), (\omega_1 E, \omega_1 E, \omega_1 E), (\omega_1^2 E, \omega_1^2 E, \omega_1^2 E)\}$, $\omega_1 = -\frac{1}{2} + \frac{\sqrt{3}}{2}e_1$.

Proof. We define a mapping $\varphi : SU(3) \times SU(3) \times SU(3) \rightarrow (E_6)^w$ by

$$\begin{aligned}\varphi(P, A, B)(X + M) &= h(A, B)Xh(A, B)^* + PM\tau h(A, B)^*, \\ X + M &\in (\mathfrak{J}_C)^C \oplus M(3, \mathbf{C})^C = \mathfrak{J}^C.\end{aligned}$$

We first have to show that $\alpha = \varphi(P, A, B) \in (E_6)^w$, that is, α leaves

$$\begin{aligned}(X + M) \times (Y + N) &= (X \times Y - \frac{1}{2}(M^*N + N^*M)) - \frac{1}{2}(MY + NX + \overline{M \times N}), \\ \langle X + M, Y + N \rangle &= \langle X, Y \rangle + 2\langle M, N \rangle\end{aligned}$$

invariant, namely, α satisfies

$$\begin{aligned}\tau\alpha\tau((X + M) \times (Y + N)) &= \alpha(X + M) \times \alpha(Y + N), \\ \langle \alpha(X + M), \alpha(Y + N) \rangle &= \langle X + M, Y + N \rangle.\end{aligned}$$

Now, for $X + M, Y + N \in (\mathfrak{J}_C)^C \oplus M(3, \mathbf{C})^C = \mathfrak{J}^C$, the relations

$$\tau\alpha\tau(X \times Y) = \alpha X \times \alpha Y, \quad \langle \alpha X, \alpha Y \rangle = \langle X, Y \rangle$$

are already shown in Proposition 3.13.4. Next,

$$\begin{aligned}(PM\tau h(A, B)^*)^*(PN\tau h(A, B)^*) &= \tau h(A, B)M^*P^*PN\tau h(A, B)^* \\ &= \tau(h(A, B)\tau(M^*N)h(A, B)^*), \\ (PM\tau h(A, B)^*)(h(A, B)Yh(A, B)^*) &= \tau(P\tau(MY)\tau h(A, B)^*), \\ \overline{(PM\tau h(A, B)^*) \times (PN\tau h(A, B)^*)} &= \overline{^t\tilde{P}(M \times N)\tau^t(h(A, B)^*)} \sim \\ &= P(\overline{M \times N})h(A, B)^* = \tau(P(\overline{M \times N})\tau h(A, B)^*).\end{aligned}$$

Furthermore, we have

$$\begin{aligned}\langle \alpha X, \alpha M \rangle &= 0 = \langle X, M \rangle, \\ \langle \alpha M, \alpha N \rangle &= (\tau\alpha M, \alpha N) = (P\tau Mh(A, B)^*, PN\tau h(A, B)^*) \\ &= (\tau M, N) = \langle M, N \rangle,\end{aligned}$$

and so $\alpha \in E_6$. Clearly, $w\alpha = \alpha w$, so that $\alpha \in (E_6)^w$. Evidently φ is a homomorphism. We shall now show that φ is onto. Let $\alpha \in (E_6)^w$. The restriction α' of α to $(\mathfrak{J}^C)_w = \{X \in \mathfrak{J}^C \mid wX = X\} = (\mathfrak{J}_C)^C$ belongs to $E_{6, \mathbf{C}}$: $\alpha' \in E_{6, \mathbf{C}}$. Hence, there exist $A, B \in SU(3)$ such that

$$\alpha X = h(A, B)Xh(A, B)^* \quad \text{or} \quad \alpha X = h(A, B)\overline{X}h(A, B)^*, \quad X \in (\mathfrak{J}_C)^C$$

(Proposition 3.13.4). In the former case, let $\beta = \varphi(E, A, B)^{-1}\alpha$, then $\beta|(\mathfrak{J}_C)^C = 1$. Hence $\beta \in G_2$, moreover $\beta \in (G_2)_{e_1} = (G_2)^w = SU(3)$ (Theorem 1.9.4). Thus there exists $P \in SU(3)$ such that

$$\beta(X + M) = X + PM = \varphi(P, E, E)(X + M), \quad X + M \in (\mathfrak{J}_C)^C \oplus M(3, \mathbf{C})^C = \mathfrak{J}^C.$$

Therefore we have

$$\alpha = \varphi(E, A, B)\beta = \varphi(E, A, B)\varphi(P, E, E) = \varphi(P, A, B).$$

In the latter case, consider the mapping $\gamma_1 : \mathfrak{J}^C \rightarrow \mathfrak{J}^C, \gamma_1(X + M) = \overline{X} + \overline{M}, X + M \in (\mathfrak{J}_C)^C \oplus M(3, \mathbf{C})^C = \mathfrak{J}^C$ and remember that $\gamma_1 \in G_2 \subset F_4 \subset E_6$. Let $\beta = \alpha^{-1}\varphi(E, A, B)\gamma_1$, then $\beta \in E_6$ and $\beta|(\mathfrak{J}_C)^C = 1$. Hence $\beta \in (G_2)_{e_1} = (G_2)^w$ (Theorem 1.9.4) $\subset (E_6)^w$. Since α and $\varphi(E, A, B) \in (E_6)^w$, we have $\gamma_1 \in (E_6)^w$, so that $\gamma_1 \in (G_2)^w$ which is a contradiction (Theorem 1.9.4). Therefor we see that φ is onto. That $\text{Ker } \varphi = \{(E, E, E), (\omega_1 E, \omega_1 E, \omega_1 E), (\omega_1^2 E, \omega_1^2 E, \omega_1^2 E)\} = \mathbf{Z}_3$ can be easily obtained. Thus we have the isomorphism $(SU(3) \times SU(3) \times SU(3))/\mathbf{Z}_3 \cong (E_6)^w$.

Remark 1. Since $(E_6)^w$ is connected as the fixed points subgroup of E_6 by an automorphism w of order 3 of the simply connected group E_6 , the fact that $\varphi : SU(3) \times SU(3) \times SU(3) \rightarrow (E_6)^w$ is onto can be proved as follows. The elements

$$\begin{aligned} &G_{01}, \quad G_{23}, \quad G_{45}, \quad G_{67}, \quad G_{26} + G_{37}, \quad -G_{27} + G_{36}, \\ &G_{24} + G_{35}, \quad -G_{25} + G_{34}, \quad G_{46} + G_{57}, \quad -G_{47} + G_{56}, \\ &\tilde{A}_1(1), \quad \tilde{A}_2(1), \quad \tilde{A}_3(1), \quad \tilde{A}_1(e_1), \quad \tilde{A}_2(e_1), \quad \tilde{A}_3(e_1), \\ &(E_1 - E_2)^\sim, \quad \tilde{F}_1(1), \quad \tilde{F}_2(1), \quad \tilde{F}_3(1), \\ &(E_2 - E_3)^\sim, \quad \tilde{F}_1(e_1), \quad \tilde{F}_2(e_1), \quad \tilde{F}_3(e_1) \end{aligned}$$

forms an \mathbf{R} -basis of $(\mathfrak{e}_6)^w$. So $\dim((\mathfrak{e}_6)^w) = 16 + 8 = 24 = 8 + 8 + 8 = \dim(\mathfrak{su}(3) \oplus \mathfrak{su}(3) \oplus \mathfrak{su}(3))$, Hence φ is onto.

Remark 2. The group E_6 has a subgroup which is isomorphic to the group $((SU(3) \times SU(3) \times SU(3))/\mathbf{Z}_3) \cdot \mathbf{Z}_2$ which is the semi-direct product of the groups $(SU(3) \times SU(3) \times SU(3))/\mathbf{Z}_3$ and \mathbf{Z}_2 (the action of $\mathbf{Z}_2 = \{1, \gamma_1\}$ on the group $SU(3) \times SU(3) \times SU(3)$ is $\gamma_1(P, A, B) = (\overline{P}, \overline{B}, \overline{A})$).

3.14. Complex exceptional Lie group E_6^C

Theorem 3.14.1. *The polar decomposition of the Lie group E_6^C is given by*

$$E_6^C \simeq E_6 \times \mathbf{R}^{78}.$$

In particular, E_6^C is a simply connected complex Lie group of type E_6 .

Proof. Evidently E_6^C is an algebraic subgroup of $\text{Iso}_C(\mathfrak{J}^C) = GL(27, \mathbf{C})$. The conjugate transposed mapping α^* of $\alpha \in E_6^C$ with respect to the inner product $\langle X, Y \rangle$ is $\alpha^* = \tau^t \alpha \tau \in E_6^C$. Hence, from Chevalley's lemma, we have

$$E_6^C \simeq (E_6^C \cap U(\mathfrak{J}^C)) \times \mathbf{R}^d = E_6 \times \mathbf{R}^d, \quad d = 78.$$

Since E_6 is simply connected (Theorem 3.9.2), E_6^C is also simply connected. The Lie algebra of the group E_6^C is \mathfrak{e}_6^C , so E_6^C is a complex simple Lie group of type E_6 .

3.15. Non-compact exceptional Lie groups $E_{6(6)}$, $E_{6(2)}$, $E_{6(-14)}$ and $E_{6(-26)}$ of type E_6

We define Hermitian inner products $\langle X, Y \rangle_\gamma$ and $\langle X, Y \rangle_\sigma$ in $\mathfrak{J}(3, \mathfrak{C}^C)$ respectively by

$$\langle X, Y \rangle_\gamma = \langle \gamma X, Y \rangle, \quad \langle X, Y \rangle_\sigma = \langle \sigma X, Y \rangle.$$

Now, we define groups $E_{6(6)}$, $E_{6(2)}$, $E_{6(-14)}$ and $E_{6(-26)}$ respectively by

$$\begin{aligned} E_{6(6)} &= \{ \alpha \in \text{Iso}_{\mathbf{R}}(\mathfrak{J}(3, \mathfrak{C}')) \mid \det(\alpha X) = \det X \}, \\ E_{6(2)} &= \{ \alpha \in \text{Iso}_C(\mathfrak{J}(3, \mathfrak{C}^C)) \mid \det(\alpha X) = \det X, \langle \alpha X, \alpha Y \rangle_\gamma = \langle X, Y \rangle_\gamma \}, \\ E_{6(-14)} &= \{ \alpha \in \text{Iso}_C(\mathfrak{J}(3, \mathfrak{C}^C)) \mid \det(\alpha X) = \det X, \langle \alpha X, \alpha Y \rangle_\sigma = \langle X, Y \rangle_\sigma \}, \\ E_{6(-26)} &= \{ \alpha \in \text{Iso}_{\mathbf{R}}(\mathfrak{J}(3, \mathfrak{C})) \mid \det(\alpha X) = \det X \}. \end{aligned}$$

These groups can also be defined by

$$E_{6(6)} \cong (E_6^C)^{\tau\gamma}, \quad E_{6(2)} \cong (E_6^C)^{\tau\lambda\gamma}, \quad E_{6(-14)} \cong (E_6^C)^{\tau\lambda\sigma}, \quad E_{6(-26)} \cong (E_6^C)^\tau.$$

Theorem 3.15.1. *The polar decompositions of the Lie groups $E_{6(6)}$, $E_{6(2)}$, $E_{6(-14)}$ and $E_{6(-26)}$ are respectively given by*

$$\begin{aligned} E_{6(6)} &\simeq Sp(4)/\mathbf{Z}_2 \times \mathbf{R}^{42}, \\ E_{6(2)} &\simeq (Sp(1) \times SU(6))/\mathbf{Z}_2 \times \mathbf{R}^{40}, \\ E_{6(-14)} &\simeq (U(1) \times Spin(10))/\mathbf{Z}_4 \times \mathbf{R}^{32}, \\ E_{6(-26)} &\simeq F_4 \times \mathbf{R}^{26}. \end{aligned}$$

Proof. These are the facts corresponding to Theorems 3.12.2, 3.11.4, 3.10.7 and 3.7.1.

Theorem 3.15.2. *The centers of the groups $E_{6(6)}$, $E_{6(2)}$, $E_{6(-14)}$ and $E_{6(-26)}$ are given by*

$$z(E_{6(6)}) = \{1\}, \quad z(E_{6(2)}) = \mathbf{Z}_3, \quad z(E_{6(-14)}) = \mathbf{Z}_3, \quad z(E_{6(-26)}) = \{1\}.$$

4. Exceptional Lie group E_7

4.1. Freudenthal vector space \mathfrak{P}^C

Definition. We define a C -vector space \mathfrak{P}^C , called the Freudenthal C -vector space, by

$$\mathfrak{P}^C = \mathfrak{J}^C \oplus \mathfrak{J}^C \oplus C \oplus C.$$

\mathfrak{P}^C is a 56 dimensional C -vector space. An element $\begin{pmatrix} X \\ Y \\ \xi \\ \eta \end{pmatrix}$ of \mathfrak{P}^C is often denoted by

(X, Y, ξ, η) or sometimes by $\dot{X} + \dot{Y} + \dot{\xi} + \dot{\eta}$. In \mathfrak{P}^C , we define an inner product (P, Q) , a positive definite Hermitian inner product $\langle P, Q \rangle$ and a skew-symmetric inner product $\{P, Q\}$ respectively by

$$\begin{aligned} (P, Q) &= (X, Z) + (Y, W) + \xi\zeta + \eta\omega, \\ \langle P, Q \rangle &= \langle X, Z \rangle + \langle Y, W \rangle + (\tau\xi)\zeta + (\tau\eta)\omega, \\ \{P, Q\} &= (X, W) - (Z, Y) + \xi\omega - \zeta\eta, \end{aligned}$$

where $P = (X, Y, \xi, \eta)$, $Q = (Z, W, \zeta, \omega) \in \mathfrak{P}^C$.

For $\phi \in \mathfrak{e}_6^C$, $A, B \in \mathfrak{J}^C$, $\nu \in C$, we define a C -linear mapping $\Phi(\phi, A, B, \nu) : \mathfrak{P}^C \rightarrow \mathfrak{P}^C$ by

$$\begin{aligned} \Phi(\phi, A, B, \nu) \begin{pmatrix} X \\ Y \\ \xi \\ \eta \end{pmatrix} &= \begin{pmatrix} \phi - \frac{1}{3}\nu & 2B & 0 & A \\ 2A & -{}^t\phi + \frac{1}{3}\nu & B & 0 \\ 0 & A & \nu & 0 \\ B & 0 & 0 & -\nu \end{pmatrix} \begin{pmatrix} X \\ Y \\ \xi \\ \eta \end{pmatrix} \\ &= \begin{pmatrix} \phi X - \frac{1}{3}\nu X + 2B \times Y + \eta A \\ 2A \times X - {}^t\phi Y + \frac{1}{3}\nu Y + \xi B \\ (A, Y) + \nu\xi \\ (B, X) - \nu\eta \end{pmatrix}. \end{aligned}$$

Definition. For $P = (X, Y, \xi, \eta)$, $Q = (Z, W, \zeta, \omega) \in \mathfrak{P}^C$, we define a C -linear mapping $P \times Q : \mathfrak{P}^C \rightarrow \mathfrak{P}^C$ by

$$P \times Q = \Phi(\phi, A, B, \nu), \quad \begin{cases} \phi = -\frac{1}{2}(X \vee W + Z \vee Y) \\ A = -\frac{1}{4}(2Y \times W - \xi Z - \zeta X) \\ B = \frac{1}{4}(2X \times Z - \eta W - \omega Y) \\ \nu = \frac{1}{8}((X, W) + (Z, Y) - 3(\xi\omega + \zeta\eta)). \end{cases}$$

Lemma 4.1.1. For $P, Q, R \in \mathfrak{P}^C$, we have

- (1) $P \times Q = Q \times P$.
- (2) $(P \times Q)P - (P \times P)Q + \frac{3}{8}\{P, Q\}P = 0$.
- (3) $(P \times R)Q - (Q \times R)P + \frac{1}{8}\{Q, R\}P - \frac{1}{8}\{P, R\}Q - \frac{1}{4}\{P, Q\}R = 0$.

Proof. (1) is evident.

(2) For $P = (X, Y, \xi, \eta)$, $Q = (Z, W, \zeta, \omega) \in \mathfrak{P}^C$, we have

$$\begin{aligned}
& (P \times Q)P \\
&= \Phi \left(-\frac{1}{2}(X \vee W + Z \vee Y), -\frac{1}{4}(2Y \times W - \xi Z - \zeta X), \frac{1}{4}(2X \times Z - \eta W - \omega Y), \right. \\
& \quad \left. \frac{1}{8}((X, W) + (Z, Y) - 3(\xi\omega + \zeta\eta)) \right) (X, Y, \xi, \eta) \\
&= \cdots (\text{using the formula } X \vee Y \text{ of Lemma 3.4.1), etc.}) \cdots \\
&= \Phi \left(-X \vee Y, -\frac{1}{2}(Y \times Y - \xi X), \frac{1}{2}(X \times X - \eta Y), \frac{1}{4}((X, Y) - 3\xi\eta) \right) (Z, W, \zeta, \omega), \\
& \quad -\frac{3}{8}((X, W) - (Z, Y) + \xi\omega - \zeta\eta)(X, Y, \xi, \eta) \\
&= (P \times P)Q - \frac{3}{8}\{P, Q\}P.
\end{aligned}$$

(3) In the equality (2), put $P + R$ in the place of P , then we have

$$2(P \times R)Q - (P \times Q)R - (R \times Q)P + \frac{3}{8}\{Q, R\}P - \frac{3}{8}\{P, Q\}R = 0. \quad (\text{i})$$

Exchanging P with Q , we see that

$$2(Q \times R)P - (Q \times P)R - (R \times P)Q + \frac{3}{8}\{P, R\}Q - \frac{3}{8}\{Q, P\}R = 0. \quad (\text{ii})$$

Taking $((\text{i}) - (\text{ii})) \div 3$, we have

$$(P \times R)Q - (Q \times R)P + \frac{1}{8}\{Q, R\}P - \frac{1}{8}\{P, R\}Q - \frac{1}{4}\{P, Q\}R = 0.$$

We define a space \mathfrak{M}^C , called the complex Freudenthal manifold, by

$$\begin{aligned}
\mathfrak{M}^C &= \{P \in \mathfrak{P}^C \mid P \times P = 0, P \neq 0\} \\
&= \left\{ \begin{array}{l} P = (X, Y, \xi, \eta) \in \mathfrak{P}^C \\ P \neq 0 \end{array} \left| \begin{array}{l} X \vee Y = 0, (X, Y) = 3\xi\eta, \\ X \times X = \eta Y, Y \times Y = \xi X \end{array} \right. \right\}.
\end{aligned}$$

Lemma 4.1.2. The following elements (assuming $\xi \neq 0, \eta \neq 0$) of \mathfrak{P}^C

$$\begin{pmatrix} X \\ \frac{1}{\eta}(X \times X) \\ \frac{1}{\eta^2}\det X \\ \eta \end{pmatrix}, \quad \begin{pmatrix} \frac{1}{\xi}(Y \times Y) \\ Y \\ \xi \\ \frac{1}{\xi^2}\det Y \end{pmatrix}, \quad \mathbf{i} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{j} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

belong to \mathfrak{M}^C .

Proof. This is clear from $X \vee (X \times X) = 0$ (Lemma 3.5.4.(1)) and $(X \times X) \times (X \times X) = (\det X)X$ (Lemma 2.1.1.(3)).

4.2. Compact exceptional Lie group E_7

Definition. We define the groups E_7^C and E_7 respectively by

$$\begin{aligned} E_7^C &= \{\alpha \in \text{Iso}_C(\mathfrak{P}^C) \mid \alpha(P \times Q)\alpha^{-1} = \alpha P \times \alpha Q\}, \\ E_7 &= \{\alpha \in \text{Iso}_C(\mathfrak{P}^C) \mid \alpha(P \times Q)\alpha^{-1} = \alpha P \times \alpha Q, \langle \alpha P, \alpha Q \rangle = \langle P, Q \rangle\}. \end{aligned}$$

E_7 is a subgroup of E_7^C .

Theorem 4.2.1. E_7 is a compact Lie group.

Proof. E_7 is a compact Lie group as a closed subgroup of the unitary group

$$U(56) = U(\mathfrak{P}^C) = \{\alpha \in \text{Iso}_C(\mathfrak{P}^C) \mid \langle \alpha P, \alpha Q \rangle = \langle P, Q \rangle\}.$$

Proposition 4.2.2. For $\alpha \in E_7^C$ (and so, for $\alpha \in E_7$), we have

- (1) $\alpha\mathfrak{M}^C = \mathfrak{M}^C$.
- (2) $\{\alpha P, \alpha Q\} = \{P, Q\}$, $P, Q \in \mathfrak{P}^C$.

Proof. (1) It is sufficient to prove that $\alpha\mathfrak{M}^C \subset \mathfrak{M}^C$. Now, for $P \in \mathfrak{M}^C$, we have $\alpha P \times \alpha P = \alpha(P \times P)\alpha^{-1} = \alpha 0 \alpha^{-1} = 0$. Hence $\alpha P \in \mathfrak{M}^C$.

$$\begin{aligned} (2) \quad \{\alpha P, \alpha Q\}\alpha P &= \frac{8}{3}((\alpha P \times \alpha P)\alpha Q - (\alpha P \times \alpha Q)\alpha P) \quad (\text{Lemma 4.1.1.(2)}) \\ &= \frac{8}{3}(\alpha(P \times P)Q - \alpha(P \times Q)P) = \{P, Q\}\alpha P. \end{aligned}$$

It follows $\{\alpha P, \alpha Q\} = \{P, Q\}$.

4.3. Lie algebra \mathfrak{e}_7 of E_7

Before we investigate the Lie algebra \mathfrak{e}_7 of the group E_7 , we shall first study the Lie algebra \mathfrak{e}_7^C of the group E_7^C .

Theorem 4.3.1. The Lie algebra \mathfrak{e}_7^C of the group E_7^C is given by

$$\mathfrak{e}_7^C = \{\Phi(\phi, A, B, \nu) \in \text{Hom}_C(\mathfrak{P}^C) \mid \phi \in \mathfrak{e}_6^C, A, B \in \mathfrak{J}^C, \nu \in C\}.$$

The Lie bracket $[\Phi_1, \Phi_2]$ in \mathfrak{e}_7^C is given by

$$[\Phi(\phi_1, A_1, B_1, \nu_1), \Phi(\phi_2, A_2, B_2, \nu_2)] = \Phi(\phi, A, B, \nu),$$

where

$$\begin{cases} \phi = [\phi_1, \phi_2] + 2A_1 \vee B_2 - 2A_2 \vee B_1 \\ A = \left(\phi_1 + \frac{2}{3}\nu_1\right)A_2 - \left(\phi_2 + \frac{2}{3}\nu_2\right)A_1 \\ B = -\left({}^t\phi_1 + \frac{2}{3}\nu_1\right)B_2 + \left({}^t\phi_2 + \frac{2}{3}\nu_2\right)B_1 \\ \nu = (A_1, B_2) - (B_1, A_2). \end{cases}$$

In particular, the dimension of \mathfrak{e}_7^C is

$$\dim_C(\mathfrak{e}_7^C) = 78 + 27 \times 2 + 1 = 133.$$

Proof. Before we show that \mathfrak{e}_7^C is the Lie algebra of the group E_7^C , we first check the form of the Lie bracket in \mathfrak{e}_7^C . For $\Phi_i \in \mathfrak{e}_7^C$ and $P \in \mathfrak{P}^C$, we have

$$\begin{aligned} [\Phi_1, \Phi_2]P &= \Phi_1\Phi_2P - \Phi_2\Phi_1P \\ &= \cdots (\text{using the formula of } A \vee B \text{ (Lemma 3.4.1) etc.}) \cdots \\ &= \Phi P. \end{aligned}$$

This Φ is that of the theorem.

We now determine the Lie algebra \mathfrak{e}_7^C of the group E_7^C . Since $\alpha \in E_7^C$ satisfies

$$\begin{cases} \alpha P \times \alpha P = 0, & P \in \mathfrak{M}^C \\ \{\alpha P, \alpha Q\} = \{P, Q\}, & P, Q \in \mathfrak{P}^C \end{cases} \quad (\text{Proposition 4.2.2.(2)}),$$

if $\Phi \in \text{Hom}_C(\mathfrak{P}^C)$ belongs to \mathfrak{e}_7^C , then Φ satisfies

$$\begin{cases} \Phi P \times P = 0, & P \in \mathfrak{M}^C & (i) \\ \{\Phi P, Q\} + \{P, \Phi Q\} = 0, & P, Q \in \mathfrak{P}^C. & (ii) \end{cases}$$

Since $\Phi \in \mathfrak{e}_7^C$ is a C -linear mapping of $\mathfrak{P}^C = \mathfrak{J}^C \oplus \mathfrak{J}^C \oplus C \oplus C$, Φ is of the form

$$\Phi = \begin{pmatrix} g & l & C & A \\ k & h & B & D \\ c & a & \nu & \lambda \\ b & d & \kappa & \mu \end{pmatrix}, \quad \begin{aligned} &g, h, k, l \in \text{Hom}_C(\mathfrak{J}^C), \\ &a, b, c, d \in \text{Hom}_C(\mathfrak{J}^C, C), \\ &A, B, C, D \in \mathfrak{J}^C, \\ &\nu, \mu, \kappa, \lambda \in C. \end{aligned}$$

For $0 \neq r \in C$, we define a C -linear isomorphism $f_r : \mathfrak{P}^C \rightarrow \mathfrak{P}^C$ by

$$f_r(X, Y, \xi, \eta) = (X, rY, r^2\xi, r^{-1}\eta),$$

then it is easy to see that f_r satisfies

$$rf_r(P \times Q)f_r^{-1} = f_rP \times f_rQ, \quad P, Q \in \mathfrak{P}^C.$$

Hence we see that for $\alpha \in E_7^C$ we have $f_r\alpha f_r^{-1} \in E_7^C$. Therefore, for $\Phi \in \mathfrak{e}_7^C$, we have

$$\mathfrak{e}_7^C \ni f_r \Phi f_r^{-1} = \begin{pmatrix} g & r^{-1}l & r^{-2}C & rA \\ rk & h & r^{-1}B & r^2D \\ r^2c & ra & \nu & r^3\lambda \\ r^{-1}b & r^{-2}d & r^{-3}k & \mu \end{pmatrix}$$

for any $0 \neq r \in C$. Hence Φ is decomposed as

$$\Phi = \Phi_{-3} + \Phi_{-2} + \Phi_{-1} + \Phi_0 + \Phi_1 + \Phi_2 + \Phi_3, \quad \Phi_i \in \mathfrak{e}_7^C,$$

where

$$\begin{aligned} \Phi_{-3} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \kappa & 0 \end{pmatrix}, & \Phi_3 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\ \Phi_{-2} &= \begin{pmatrix} 0 & 0 & C & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & d & 0 & 0 \end{pmatrix}, & \Phi_2 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & D \\ c & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\ \Phi_{-1} &= \begin{pmatrix} 0 & l & 0 & 0 \\ 0 & 0 & B & 0 \\ 0 & 0 & 0 & 0 \\ b & 0 & 0 & 0 \end{pmatrix}, & \Phi_1 &= \begin{pmatrix} 0 & 0 & 0 & A \\ k & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\ \Phi_0 &= \begin{pmatrix} g & 0 & 0 & 0 \\ 0 & h & 0 & 0 \\ 0 & 0 & \nu & 0 \\ 0 & 0 & 0 & \mu \end{pmatrix}. \end{aligned}$$

The relation $\Phi_{-3}(0, 0, 1, 0) \times (0, 0, 1, 0) = 0$ implies $\kappa = 0$, hence $\Phi_{-3} = 0$. Similarly $\Phi_3 = 0$. The relation $\Phi_{-2}(0, 0, 1, 0) \times (0, 0, 1, 0) = 0$ implies $C = 0$. Moreover, the relation $\Phi_{-2}P \times P = 0$ for $P = (Y \times Y, Y, 1, \det Y) \in \mathfrak{M}^C$, that is, $(0, 0, 0, d(Y)) \times (Y \times Y, Y, 1, \det Y) = 0$ implies $d = 0$. Hence $\Phi_{-2} = 0$. Similarly $\Phi_2 = 0$. The relation $\Phi_{-1}P \times P = 0$ for $P = (Y \times Y, Y, 1, \det Y) \in \mathfrak{M}^C$, that is, $(l(Y), B, 0, b(Y \times Y)) \times (Y \times Y, Y, 1, \det Y) = 0$ implies

$$l(Y) = 2B \times Y, \quad Y \in \mathfrak{J}^C.$$

Next, the relation $\Phi_{-1}P \times P = 0$ for $P = (X, X \times X, \det X, 1) \in \mathfrak{M}^C$, that is, $(l(X \times X), (\det X)B, 0, b(X)) \times (X, X \times X, \det X, 1) = 0$, (from the 4-th condition) we have

$$2(B \times (X \times X), X \times X) + (\det X)(X, B) = 3(\det X)b(X),$$

hence, $3(\det X)(B, X) = 3(\det X)b(X)$, and so we have

$$b(X) = (B, X), \quad X \in \mathfrak{J}^C.$$

(Since b is continuous, the above is also valid for $X \in \mathfrak{J}^C$ such that $\det X = 0$). Similarly, using Φ_1 , we have

$$k(X) = 2A \times X, \quad a(Y) = (A, Y), \quad X, Y \in \mathfrak{J}^C.$$

The relation $\Phi_0 P \times P = 0$ for $P = (X, X \times X, \det X, 1) \in \mathfrak{M}^C$, that is, $(g(X), h(X \times X), (\det X)\nu, \mu) \times (X, X \times X, \det X, 1) = 0$ implies

$$2g(X) \times X = \mu X \times X + h(X \times X), \quad (\text{i})$$

$$2h(X \times X) \times (X \times X) = (\det X)(\nu X + g(X)), \quad (\text{ii})$$

$$(g(X), X \times X) + (h(X \times X), X) = 3(\nu + \mu)\det X. \quad (\text{iii})$$

Putting $\phi = g - \frac{1}{3}(\nu + 2\mu)1$, then, using (i) and (iii), we have

$$\begin{aligned} 3(\phi X, X, X) &= 3(g(X), X, X) - (\nu + 2\mu)(X, X, X) \\ &= (\mu X \times X + h(X \times X), X) + (g(X), X \times X) - 3(\nu + 2\mu)\det X \\ &= 3\mu\det X + 3(\nu + \mu)\det X - 3(\nu + 2\mu)\det X = 0. \end{aligned}$$

Therefore

$$\phi \in \mathfrak{e}_6^C.$$

Similarly $\psi = h - \frac{1}{3}(2\nu + \mu)1 \in \mathfrak{e}_6^C$. Furthermore, from (ii), we have

$$\begin{aligned} 2\left(\psi(X \times X) + \frac{1}{3}(2\nu + \mu)(X \times X)\right) \times (X \times X) \\ = (\det X)\left(\nu X + \phi X + \frac{1}{3}(\nu + 2\mu)X\right), \end{aligned}$$

so $2\psi(X \times X) \times (X \times X) = (\det X)\phi X$, and so (for a while, instead of $-^t\psi$, we use ψ' again) $\psi'((X \times X) \times (X \times X)) = (\det X)\phi X$ (Lemma 3.4.3.(1)), hence $(\det X)\psi'X = (\det X)\phi X$. Therefore, we have $\psi'X = \phi X, X \in \mathfrak{J}^C$, (even if for $X \in \mathfrak{J}^C$ such that $\det X = 0$), that is,

$$\psi' = \phi.$$

Finally, the relation $\{\Phi(0, 0, 1, 0), (0, 0, 0, 1)\} + \{(0, 0, 1, 0), \Phi(0, 0, 0, 1)\} = 0$ implies $\nu + \mu = 0$. Thus we see that $\Phi \in \mathfrak{e}_7^C$ is of the form

$$\begin{aligned} \Phi &= \begin{pmatrix} \phi - \frac{1}{3}\nu & 2B & 0 & A \\ 2A & \phi' + \frac{1}{3}\nu & B & 0 \\ 0 & A & \nu & 0 \\ B & 0 & 0 & -\nu \end{pmatrix} \\ &= \Phi(\phi, A, B, \nu), \quad \phi \in \mathfrak{e}_6^C, A, B \in \mathfrak{J}^C, \nu \in C. \end{aligned}$$

Conversely, we shall show that $\Phi = \Phi(\phi, A, B, \nu)$, $\phi \in \mathfrak{e}_6^C$, $A, B \in \mathfrak{J}^C$, $\nu \in C$ belongs to \mathfrak{e}_7^C , that is, $\exp t\Phi \in E_7^C$ for all $t \in C$. For this purpose, we prove the following proposition.

Proposition 4.3.2. For $\Phi = \Phi(\phi, A, B, \nu)$, $\phi \in \mathfrak{e}_6^C$, $A, B \in \mathfrak{J}^C$, $\nu \in C$ and $P, Q \in \mathfrak{P}^C$, we have

$$[\Phi, P \times Q] = \Phi P \times Q + P \times \Phi Q.$$

Proof. It is sufficient to show that $[\Phi, P \times P] = 2\Phi P \times P$. For $P = (X, Y, \xi, \eta) \in \mathfrak{P}^C$, we have

$$\begin{aligned} & [\Phi, P \times P] \\ &= \left[\Phi(\phi, A, B, \nu), \Phi\left(-X \vee Y, -\frac{1}{2}(Y \times Y - \xi X), \frac{1}{2}(X \times X - \eta Y), \frac{1}{4}((X, Y) - 3\xi\eta)\right) \right] \\ &= \cdots (\text{using } \phi(X \times Y) = \phi'X \times Y + X \times \phi'Y \text{ (Lemma 3.4.3.(1)), } [\phi, A \vee B] \\ &\quad = \phi A \vee B + A \vee \phi' B \text{ (Lemma 3.4.4.(1)), the formula about } A \vee B \text{ (Lemma 3.4.1) etc.}) \cdots \\ &= 2\Phi P \times P. \end{aligned}$$

We will now return to the proof of Theorem 4.3.1. For $\Phi = \Phi(\phi, A, B, \nu)$, $\phi \in \mathfrak{e}_6^C$, $A, B \in \mathfrak{J}^C$, $\nu \in C$ and $t \in C$, we have

$$\begin{aligned} & (\exp t\Phi)(P \times Q)(\exp t\Phi)^{-1} \\ &= (\exp t(\text{ad}\Phi))(P \times Q) \quad ((\text{ad}\Phi)\Phi_1 = [\Phi, \Phi_1], \Phi_1 \in \mathfrak{e}_7^C) \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} (t(\text{ad}\Phi))^n (P \times Q) \\ &= \sum_{n=0}^{\infty} \frac{t^n}{n!} \left(\sum_{k+l=n} \frac{n!}{k!l!} \Phi^k P \times \Phi^l Q \right) \quad (\text{Proposition 4.3.2}) \\ &= \sum_{n=0}^{\infty} \left(\sum_{k+l=n} \frac{t^k t^l}{k!l!} \Phi^k P \times \Phi^l Q \right) \\ &= \left(\sum_{k=0}^{\infty} \frac{1}{k!} (t\Phi)^k P \right) \times \left(\sum_{l=0}^{\infty} \frac{1}{l!} (t\Phi)^l Q \right) \\ &= (\exp t\Phi)P \times (\exp t\Phi)Q. \end{aligned}$$

Hence $\exp t\Phi \in E_7^C$, so that $\Phi \in \mathfrak{e}_7^C$. Thus the proof of Theorem 4.3.1 is completed.

Definition. We define a C -linear transformation λ of \mathfrak{P}^C by

$$\lambda(X, Y, \xi, \eta) = (Y, -X, \eta, -\xi).$$

For $\alpha \in \text{Hom}_C(\mathfrak{P}^C)$, we also denote by ${}^t\alpha$ the transpose of α with respect to the inner product (P, Q) : $({}^t\alpha P, Q) = (P, \alpha Q)$.

Lemma 4.3.3. (1) $\lambda \in E_7$ and satisfies $\lambda^2 = -1$.

(2) For $P, Q \in \mathfrak{P}^C$, we have

$$(P, Q) = \{\lambda P, Q\} = -\{P, \lambda Q\}, \quad \langle P, Q \rangle = \{\tau \lambda P, Q\}.$$

(3) For $\alpha \in E_7^C$, we have

$${}^t\alpha^{-1} = \lambda\alpha\lambda^{-1}.$$

(4) For $\alpha \in E_7^C$, we have

$$\alpha \in E_7 \quad \text{if and only if} \quad \tau\lambda\alpha = \alpha\tau\lambda.$$

(5) For $\Phi(\phi, A, B, \nu) \in \mathfrak{e}_7^C$, we have

$$\lambda\Phi(\phi, A, B, \nu)\lambda^{-1} = \Phi(-{}^t\phi, -B, -A, -\nu).$$

Proof. (1) and (2) are evident.

(3) $(P, \lambda Q) = \{P, Q\} = \{\alpha P, \alpha Q\}$ (Proposition 4.2.2.(2)) $= (\alpha P, \lambda\alpha Q)$ (Lemma 4.3.3) $= (P, {}^t\alpha\lambda\alpha Q)$ implies $\lambda = {}^t\alpha\lambda\alpha$, hence ${}^t\alpha^{-1} = \lambda\alpha\lambda^{-1}$.

(4) If $\alpha \in E_7^C$ satisfies $\tau\lambda\alpha = \alpha\tau\lambda$, then $\langle \alpha P, \alpha Q \rangle = \{\tau\lambda\alpha P, \alpha Q\} = \{\alpha\tau\lambda P, \alpha Q\} = \{\tau\lambda P, Q\}$ (Proposition 4.2.2.(2)) $= \langle P, Q \rangle$ (Lemma 4.3.3), $P, Q \in \mathfrak{P}^C$. Hence, $\alpha \in E_7$. The converse implication can also be easily proved.

(5) is easily checked by direct calculations.

Theorem 4.3.4. *The Lie algebra \mathfrak{e}_7 of the group E_7 is given by*

$$\mathfrak{e}_7 = \{\Phi(\phi, A, -\tau A, \nu) \mid \phi \in \mathfrak{e}_6, A \in \mathfrak{J}^C, \nu \in i\mathbf{R}\},$$

where

$$\Phi(\phi, A, -\tau A, \nu) = \begin{pmatrix} \phi - \frac{1}{3}\nu & -2\tau A & 0 & A \\ 2A & \tau\phi\tau + \frac{1}{3}\nu & -\tau A & 0 \\ 0 & A & \nu & 0 \\ -\tau A & 0 & 0 & -\nu \end{pmatrix}.$$

The Lie bracket $[\Phi_1, \Phi_2]$ in \mathfrak{e}_7 is given by

$$[\Phi(\phi_1, A_1, -\tau A_1, \nu_1), \Phi(\phi_2, A_2, -\tau A_2, \nu_2)] = \Phi(\phi, A, -\tau A, \nu),$$

where

$$\begin{cases} \phi = [\phi_1, \phi_2] - 2A_1 \vee \tau A_2 + 2A_2 \vee \tau A_1 \\ A = \left(\phi_1 + \frac{2}{3}\nu_1\right)A_2 - \left(\phi_2 + \frac{2}{3}\nu_2\right)A_1 \\ \nu = \langle A_1, A_2 \rangle - \langle A_2, A_1 \rangle. \end{cases}$$

In particular, the dimension of \mathfrak{e}_7 is

$$\dim(\mathfrak{e}_7) = 78 + 54 + 1 = 133.$$

Proof. For $\Phi \in \mathfrak{e}_7^C$,

$$\Phi \in \mathfrak{e}_7 \quad \text{if and only if} \quad \tau\lambda\Phi\lambda^{-1}\tau = \Phi.$$

Therefore the theorem follows from Theorem 4.3.1, Lemma 4.3.3.(5) and $\tau\Phi(\phi, A, B, \nu)\tau = \Phi(\tau\phi\tau, \tau A, \tau B, \tau\nu)$.

Proposition 4.3.5. *The complexification of the Lie algebra \mathfrak{e}_7 is \mathfrak{e}_7^C .*

Proof. For $\Phi \in \mathfrak{e}_7^C$, the conjugate transposed mapping Φ^* of Φ with respect to the inner product $\langle P, Q \rangle$ of \mathfrak{P}^C is $\Phi^* = \tau\lambda\Phi\lambda\tau \in \mathfrak{e}_7^C$, and for $\Phi \in \mathfrak{e}_7^C$, Φ belongs to \mathfrak{e}_7 if and only if $\Phi^* = -\Phi$. Now, any element $\Phi \in \mathfrak{e}_7^C$ is represented as

$$\Phi = \frac{\Phi - \Phi^*}{2} + i\frac{\Phi + \Phi^*}{2i}, \quad \frac{\Phi - \Phi^*}{2}, \frac{\Phi + \Phi^*}{2i} \in \mathfrak{e}_7.$$

Hence \mathfrak{e}_7^C is the complexification of \mathfrak{e}_7 .

4.4. Simplicity of \mathfrak{e}_7^C

Theorem 4.4.1. *The Lie algebra \mathfrak{e}_7^C is simple and so \mathfrak{e}_7 is also simple.*

Proof. We use the decomposition of \mathfrak{e}_7^C of Theorem 4.3.1

$$\mathfrak{e}_7^C = \mathfrak{e}_6^C \oplus \mathfrak{N}^C,$$

where $\mathfrak{e}_6^C = \{\Phi(\phi, 0, 0, 0) \in \mathfrak{e}_7^C \mid \phi \in \mathfrak{e}_6^C\}$ and $\mathfrak{N}^C = \{\Phi(0, A, B, \nu) \in \mathfrak{e}_7^C \mid A, B \in \mathfrak{J}^C, \nu \in C\}$. Let $p : \mathfrak{e}_7^C \rightarrow \mathfrak{e}_6^C$ and $q : \mathfrak{e}_7^C \rightarrow \mathfrak{N}^C$ be projections of $\mathfrak{e}_7^C = \mathfrak{e}_6^C \oplus \mathfrak{N}^C$. Now, let \mathfrak{a} be a non-zero ideal of \mathfrak{e}_7^C . Then $p(\mathfrak{a})$ is an ideal of \mathfrak{e}_6^C . Indeed, if $\phi \in p(\mathfrak{a})$, then there exists $\Phi(0, A, B, \nu) \in \mathfrak{N}^C$ such that $\Phi(\phi, A, B, \nu) \in \mathfrak{a}$. For any $\phi_1 \in \mathfrak{e}_6^C$, we have

$$\mathfrak{a} \ni [\Phi(\phi_1, 0, 0, 0), \Phi(\phi, A, B, \nu)] = \Phi([\phi_1, \phi], \phi_1 A, \phi_1' B, 0)$$

(Theorem 4.3.1), hence $[\phi_1, \phi] \in p(\mathfrak{a})$.

We shall show that either $\mathfrak{e}_6^C \cap \mathfrak{a} \neq \{0\}$ or $\mathfrak{N}^C \cap \mathfrak{a} \neq \{0\}$. Assume that $\mathfrak{e}_6^C \cap \mathfrak{a} = \{0\}$ and $\mathfrak{N}^C \cap \mathfrak{a} = \{0\}$. Then the mapping $p|_{\mathfrak{a}} : \mathfrak{a} \rightarrow \mathfrak{e}_6^C$ is injective because $\mathfrak{N}^C \cap \mathfrak{a} = \{0\}$. Since $p(\mathfrak{a})$ is a non-zero ideal of \mathfrak{e}_6^C and \mathfrak{e}_6^C is simple, we have $p(\mathfrak{a}) = \mathfrak{e}_6^C$. Hence $\dim_C(\mathfrak{a}) = \dim_C(p(\mathfrak{a})) = \dim_C(\mathfrak{e}_6^C) = 78$. On the other hand, since $\mathfrak{e}_6^C \cap \mathfrak{a} = \{0\}$, $q|_{\mathfrak{a}} : \mathfrak{a} \rightarrow \mathfrak{N}^C$ is also injective. Hence we have $\dim_C(\mathfrak{a}) \leq \dim_C(\mathfrak{N}^C) = 27+27+1 = 55$. This leads to a contradiction.

We now consider the following two cases.

(1) Case $\mathfrak{e}_6^C \cap \mathfrak{a} \neq \{0\}$. From the simplicity of \mathfrak{e}_6^C , we have $\mathfrak{e}_6^C \cap \mathfrak{a} = \mathfrak{e}_6^C$, hence $\mathfrak{a} \supset \mathfrak{e}_6^C$. On the other hand, we have

$$\begin{aligned} \mathfrak{a} \supset [\mathfrak{a}, \mathfrak{e}_7^C] &\supset [\Phi(\mathfrak{e}_6^C, 0, 0, 0), \Phi(0, \mathfrak{J}^C, 0, 0)] \\ &= \Phi(0, \mathfrak{e}_6^C \mathfrak{J}^C, 0, 0) = \Phi(0, \mathfrak{J}^C, 0, 0) \text{ (Proposition 3.3.2.(3)).} \end{aligned}$$

Similarly we have $\Phi(0, 0, \mathfrak{J}^C, 0) \subset \mathfrak{a}$. Moreover, from

$$\mathfrak{a} \ni [\Phi(0, E_1, 0, 0), \Phi(0, 0, E_1, 0)] = \Phi(2E_1 \vee E_1, 0, 0, 1),$$

we have $\Phi(0, 0, 0, 1) \in \mathfrak{a}$, and so $\mathfrak{a} \supset \mathfrak{N}^C$. Hence $\mathfrak{a} \supset \mathfrak{e}_6^C \oplus \mathfrak{N}^C = \mathfrak{e}_7^C$.

(2) Case $\mathfrak{N}^C \cap \mathfrak{a} \neq \{0\}$. Let $\Phi(0, A, B, \nu)$ be a non-zero element of $\mathfrak{N}^C \cap \mathfrak{a}$.

(i) Case $\Phi(0, A, B, \nu), A \neq 0$. Choose $B_1 \in \mathfrak{J}^C$ such that $A \vee B_1 \neq 0$ (Lemma 3.5.4.(2)), and choose $\phi \in \mathfrak{e}_6^C$ such that $[A \vee B_1, \phi] \neq 0$ (since \mathfrak{e}_6^C is simple, such a ϕ exists because the center of \mathfrak{e}_6^C is zero). Now, we have

$$\begin{aligned}\mathfrak{a} &\ni \left[\Phi(0, A, B, \nu), \Phi\left(0, 0, 0, -\frac{3}{2}\right) \right] = \Phi(0, A, -B, 0), \\ \mathfrak{a} &\ni [\Phi(0, A, -B, 0), \Phi(0, 0, B_1, 0)] = \Phi(2A \vee B_1, 0, 0, (A, B_1)), \\ \mathfrak{a} &\ni [\Phi(2A \vee B_1, 0, 0, (A, B_1)), \Phi(\phi, 0, 0, 0)] = \Phi(2[A \vee B_1 \cdot \phi], 0, 0, 0),\end{aligned}$$

Hence this case is reduced to the case (1).

(ii) Case $\Phi(0, A, B, \nu), B \neq 0$. This case is also reduced to the case (1) in a similar way to (i).

(iii) Case $\Phi(0, 0, 0, \nu), \nu \neq 0$. If we choose $0 \neq A \in \mathfrak{J}^C$, then we have

$$\mathfrak{a} \ni [\Phi(0, 0, 0, \nu), \Phi(0, A, 0, 0)] = \Phi\left(0, \frac{2}{3}\nu A, 0, 0\right).$$

Hence this case is also reduced to the case (1).

Thus we have $\mathfrak{a} = \mathfrak{e}_7^C$.

Proposition 4.4.2. (1) \mathfrak{P}^C is an \mathfrak{e}_7^C -irreducible C -module.

$$(2) \mathfrak{e}_7^C \mathfrak{P}^C = \left\{ \sum_k \Phi_k P_k \mid \Phi_k \in \mathfrak{e}_7^C, P_k \in \mathfrak{P}^C \right\} = \mathfrak{P}^C.$$

Proof. (1) Let W be a non-zero \mathfrak{e}_7^C -invariant C -submodule of \mathfrak{P}^C . We first prove that if $(0, 0, 0, 1) \in W$, then we have $W = \mathfrak{P}^C$. Indeed,

$$\begin{aligned}W &\ni \Phi(0, X, 0, 0)(0, 0, 0, 1) = (X, 0, 0, 0), \\ W &\ni \Phi(0, E_2, 0, 0)(E_3, 0, 0, 0) = (0, E_1, 0, 0), \\ W &\ni \Phi(0, E_1, 0, 0)(0, E_1, 0, 0) = (0, 0, 1, 0), \\ W &\ni \Phi(0, 0, Y, 0)(0, 0, 1, 0) = (0, Y, 0, 0).\end{aligned}$$

Hence we have $W = \mathfrak{P}^C$. Now, let $P = (X, Y, \xi, \eta)$ be a non-zero element of W .

(i) Case $W \ni P = (X, Y, \xi, \eta), X \neq 0$. Then we have (a)

$$W \ni \Phi(0, 0, 0, 3)(X, Y, \xi, \eta) = (-X, Y, 3\xi, -3\eta), \quad (b)$$

$$W \ni \Phi(0, 0, 0, 3)(-X, Y, 3\xi, -3\eta) = (X, Y, 9\xi, 9\eta). \quad (c)$$

Taking $((a)-(b)) \div 2$, $((a)-(c)) \div 8$, we have $(X, 0, -\xi, 2\eta) \in W$, $(0, 0, \xi, \eta) \in W$, respectively. Consequently $(X, 0, 0, 3\eta) \in W$. Next, if we choose $X_1 \in \mathfrak{J}^C$ such that $(X_1, X) \neq 0$, from

$$W \ni \Phi(0, 0, X_1, 0)(X, 0, 0, 3\eta) = (0, 0, 0, (X_1, X)),$$

we have $(0, 0, 0, 1) \in W$. Hence this case is reduced to the first case.

(ii) Case $P = (0, Y, \xi, \eta), Y \neq 0$. If we choose $B \in \mathfrak{J}^C$ such that $B \times Y \neq 0$, from

$$W \ni \Phi(0, 0, B, 0)(0, Y, \xi, \eta) = (2B \times Y, \xi B, 0, 0).$$

Hence this case is reduced to the case (i).

(iii) Case $P = (0, 0, \xi, \eta), \xi \neq 0$. For $0 \neq B \in \mathfrak{J}^C$, we have

$$W \ni \Phi(0, 0, B, 0)(0, 0, \xi, \eta) = (0, \xi B, 0, 0).$$

Hence this case is also reduced to the case (ii).

Thus we have $W = \mathfrak{P}^C$.

(2) Since $\mathfrak{e}_7^C \mathfrak{P}^C$ is an \mathfrak{e}_7^C -invariant C -submodule of \mathfrak{P}^C , $\mathfrak{e}_7^C \mathfrak{P}^C = \mathfrak{P}^C$ follows from the irreducibility of \mathfrak{P}^C (above (1)).

Lemma 4.4.3. Any element $\Phi \in \mathfrak{e}_7^C$ is expressed by $\Phi = \sum_i (P_i \times Q_i)$, $P_i, Q_i \in \mathfrak{P}^C$.

Proof. Since $[\Phi, P \times Q] = \Phi P \times Q + P \times \Phi Q$ (Proposition 4.3.2), $\mathfrak{a} = \left\{ \sum_i (P_i \times Q_i) \mid P_i, Q_i \in \mathfrak{P}^C \right\}$ is an ideal of \mathfrak{e}_7^C . From the simplicity of \mathfrak{e}_7^C (Theorem 4.4.1), we have $\mathfrak{a} = \mathfrak{e}_7^C$.

4.5. Killing form of \mathfrak{e}_7^C

Definition. We define a symmetric inner product $(\Phi_1, \Phi_2)_7$ in \mathfrak{e}_7^C by

$$(\Phi_1, \Phi_2)_7 = -2(\phi_1, \phi_2)_6 - 4(A_1, B_2) - 4(A_2, B_1) - \frac{8}{3}\nu_1\nu_2,$$

where $\Phi_i = \Phi(\phi_i, A_i, B_i, \nu_i) \in \mathfrak{e}_7^C$.

Lemma 4.5.1. (1) The inner product $(\Phi_1, \Phi_2)_7$ of \mathfrak{e}_7^C is \mathfrak{e}_7^C -adjoint invariant:

$$([\Phi, \Phi_1], \Phi_2)_7 + (\Phi_1, [\Phi, \Phi_2])_7 = 0, \quad \Phi, \Phi_i \in \mathfrak{e}_7^C.$$

(2) For $\Phi \in \mathfrak{e}_7^C, P, Q \in \mathfrak{P}^C$, we have

$$(\Phi, P \times Q)_7 = \{\Phi P, Q\}.$$

Proof. (1) $([\Phi, \Phi_1], \Phi_2)_7$

$$= \left(\Phi \begin{pmatrix} [\phi, \phi_1] + 2A \vee B_1 - 2A_1 \vee B \\ \left(\phi + \frac{2}{3}\nu\right)A_1 - \left(\phi_1 + \frac{2}{3}\nu_1\right)A \\ -\left({}^t\phi + \frac{2}{3}\nu\right)B_1 + \left({}^t\phi_1 + \frac{2}{3}\nu_1\right)B \\ (A, B_1) - (B, A_1) \end{pmatrix}, \Phi \begin{pmatrix} \phi_2 \\ A_2 \\ B_2 \\ \nu_2 \end{pmatrix} \right)_7$$

$$\begin{aligned}
&= \cdots (\text{using } ([\phi, A \vee B] = \phi A \vee B + A \vee \phi' B \text{ (Lemma 3.4.4) etc.}) \cdots \\
&= -(\Phi_1, [\Phi, \Phi_2])_7.
\end{aligned}$$

(2) For $P = (X, Y, \xi, \eta), Q = (Z, W, \zeta, \omega) \in \mathfrak{P}^C$, we have

$$\begin{aligned}
(\Phi, P \times Q)_7 &= \left(\Phi \begin{pmatrix} \phi \\ A \\ B \\ \nu \end{pmatrix}, \Phi \begin{pmatrix} -\frac{1}{2}(X \vee W + Z \vee Y) \\ -\frac{1}{4}(2Y \times W - \xi Z - \zeta X) \\ \frac{1}{4}(2X \times Z - \eta W - \omega Y) \\ \frac{1}{8}((X, W) + (Z, Y) - 3(\xi\omega + \zeta\eta)) \end{pmatrix} \right)_7 \\
&= (\phi, X \vee W + Z \vee Y)_6 - (A, 2X \times Z - \eta W - \omega Y) + (2Y \times W - \xi Z - \zeta X, B) \\
&\quad - \frac{1}{3}\nu((X, W) + (Z, Y) - 3(\xi\omega + \zeta\eta)) \\
&= (\phi X, W) + (\phi Z, Y) - 2(A, X, Z) + \eta(A, W) + \omega(A, Y) + 2(Y, W, B) \\
&\quad - \xi(Z, B) - \zeta(X, B) - \frac{1}{3}\nu(X, W) - \frac{1}{3}\nu(Z, Y) + \nu(\xi\omega + \zeta\eta) \\
&= \left\{ \begin{pmatrix} \phi X - \frac{1}{3}\nu X + 2B \times Y + \eta A \\ 2A \times X - {}^t\phi Y + \frac{1}{3}\nu Y + \xi B \\ (A, Y) + \nu\xi \\ (B, X) - \nu\eta \end{pmatrix}, \begin{pmatrix} Z \\ W \\ \zeta \\ \omega \end{pmatrix} \right\} = \{\Phi P, Q\}.
\end{aligned}$$

Theorem 4.5.2. *The Killing form B_7 of the Lie algebra \mathfrak{e}_7^C is given by*

$$\begin{aligned}
B_7(\Phi_1, \Phi_2) &= -9(\Phi_1, \Phi_2)_7 \\
&= 18(\phi_1, \phi_2)_6 + 36(A_1, B_2) + 36(A_2, B_1) + 24\nu_1\nu_2 \\
&= \frac{3}{2}B_6(\phi_1, \phi_2) + 36(A_1, B_2) + 36(A_2, B_1) + 24\nu_1\nu_2 \\
&= 3\text{tr}(\Phi_1\Phi_2),
\end{aligned}$$

where $\Phi_i = \Phi(\phi_i, A_i, B_i, \nu_i) \in \mathfrak{e}_7^C$ and B_6 is the Killing form of \mathfrak{e}_6^C .

Proof. Since \mathfrak{e}_7^C is simple (Theorem 4.4.1), there exist $k, k' \in C$ such that

$$B_7(\Phi_1, \Phi_2) = k(\Phi_1, \Phi_2)_7 = k'\text{tr}(\Phi_1\Phi_2).$$

To determine these k, k' , let $\Phi_0 = \Phi_1 = \Phi_2 = \Phi(0, 0, 0, 1)$. Then we have

$$(\Phi_0, \Phi_0)_7 = -\frac{8}{3}.$$

On the other hand, we have

$$[\Phi_0, [\Phi_0, \Phi(\phi, A, B, \nu)]] = \left[\Phi_0, \Phi\left(0, \frac{2}{3}A, -\frac{2}{3}B, 0\right) \right] = \Phi\left(0, \frac{4}{9}A, \frac{4}{9}B, 0\right).$$

Hence

$$B_7(\Phi_0, \Phi_0) = \text{tr}((\text{ad}\Phi_0)^2) = \frac{4}{9} \times 27 \times 2 = 24.$$

Therefore $k = -9$. Next, from

$$\Phi_0 \Phi_0(X, Y, \xi, \eta) = \Phi_0\left(-\frac{X}{3}, \frac{Y}{3}, \xi, -\eta\right) = \left(\frac{X}{9}, \frac{Y}{9}, \xi, \eta\right),$$

we have

$$\text{tr}(\Phi_0^2) = \frac{1}{9} \times 27 \times 2 + 1 + 1 = 8.$$

Therefore $k' = 3$.

Lemma 4.5.3. *For $P \in \mathfrak{P}^C$, $P \neq 0$, there exists $Q \in \mathfrak{P}^C$ such that $P \times Q \neq 0$.*

Proof. Assume that $P \times Q = 0$ for all $Q \in \mathfrak{P}^C$. Then for any $\Phi \in \mathfrak{e}_7^C$, $0 = (\Phi, P \times Q)_7 = (\Phi, Q \times P)_7 = \{\Phi Q, P\}$ (Lemma 4.5.1.(2)). Since $\mathfrak{e}_7^C \mathfrak{P}^C = \mathfrak{P}^C$ (Proposition 4.4.2.(2)), we have $\{\mathfrak{P}^C, P\} = 0$, so that $P = 0$.

4.6. Roots of \mathfrak{e}_7^C

Theorem 4.6.1 *The rank of the Lie algebra \mathfrak{e}_7^C is 7. The roots of \mathfrak{e}_7^C relative to some Cartan subalgebra \mathfrak{h} are given by*

$$\begin{aligned} & \pm(\lambda_k - \lambda_l), \quad \pm(\lambda_k + \lambda_l), \quad 0 \leq k < l \leq 3, \\ & \pm\lambda_k \pm \frac{1}{2}(\mu_2 - \mu_3), \quad 0 \leq k \leq 3, \\ & \pm\frac{1}{2}(-\lambda_0 - \lambda_1 + \lambda_2 - \lambda_3) \pm \frac{1}{2}(\mu_3 - \mu_1), \\ & \pm\frac{1}{2}(\lambda_0 + \lambda_1 + \lambda_2 - \lambda_3) \pm \frac{1}{2}(\mu_3 - \mu_1), \\ & \pm\frac{1}{2}(-\lambda_0 + \lambda_1 + \lambda_2 + \lambda_3) \pm \frac{1}{2}(\mu_3 - \mu_1), \\ & \pm\frac{1}{2}(\lambda_0 - \lambda_1 + \lambda_2 + \lambda_3) \pm \frac{1}{2}(\mu_3 - \mu_1), \\ & \pm\frac{1}{2}(\lambda_0 - \lambda_1 + \lambda_2 - \lambda_3) \pm \frac{1}{2}(\mu_1 - \mu_2), \\ & \pm\frac{1}{2}(-\lambda_0 + \lambda_1 + \lambda_2 - \lambda_3) \pm \frac{1}{2}(\mu_1 - \mu_2), \\ & \pm\frac{1}{2}(\lambda_0 + \lambda_1 + \lambda_2 + \lambda_3) \pm \frac{1}{2}(\mu_1 - \mu_2), \\ & \pm\frac{1}{2}(-\lambda_0 - \lambda_1 + \lambda_2 + \lambda_3) \pm \frac{1}{2}(\mu_1 - \mu_2), \\ & \pm\left(\mu_j + \frac{2}{3}\nu\right), \quad 0 \leq j \leq 3, \\ & \pm\lambda_k \pm \left(\frac{1}{2}\mu_1 - \frac{2}{3}\nu\right), \quad 0 \leq k \leq 3, \\ & \pm\frac{1}{2}(-\lambda_0 - \lambda_1 + \lambda_2 - \lambda_3) \pm \left(\frac{1}{2}\mu_2 - \frac{2}{3}\nu\right), \\ & \pm\frac{1}{2}(\lambda_0 + \lambda_1 + \lambda_2 - \lambda_3) \pm \left(\frac{1}{2}\mu_2 - \frac{2}{3}\nu\right), \end{aligned}$$

$$\begin{aligned}
& \pm \frac{1}{2}(-\lambda_0 + \lambda_1 + \lambda_2 + \lambda_3) \pm \left(\frac{1}{2}\mu_2 - \frac{2}{3}\nu\right), \\
& \pm \frac{1}{2}(\lambda_0 - \lambda_1 + \lambda_2 + \lambda_3) \pm \left(\frac{1}{2}\mu_2 - \frac{2}{3}\nu\right), \\
& \pm \frac{1}{2}(\lambda_0 - \lambda_1 + \lambda_2 - \lambda_3) \pm \left(\frac{1}{2}\mu_3 - \frac{2}{3}\nu\right), \\
& \pm \frac{1}{2}(-\lambda_0 + \lambda_1 + \lambda_2 - \lambda_3) \pm \left(\frac{1}{2}\mu_3 - \frac{2}{3}\nu\right), \\
& \pm \frac{1}{2}(\lambda_0 + \lambda_1 + \lambda_2 + \lambda_3) \pm \left(\frac{1}{2}\mu_3 - \frac{2}{3}\nu\right), \\
& \pm \frac{1}{2}(-\lambda_0 - \lambda_1 + \lambda_2 + \lambda_3) \pm \left(\frac{1}{2}\mu_3 - \frac{2}{3}\nu\right)
\end{aligned}$$

with $\mu_1 + \mu_2 + \mu_3 = 0$.

Proof. We use the decomposition of \mathfrak{e}_7^C in Theorem 4.3.1

$$\mathfrak{e}_7^C = \mathfrak{e}_6^C \oplus \mathfrak{J}^C \oplus \mathfrak{J}^C \oplus C.$$

Let

$$\mathfrak{h} = \left\{ \Phi\left(\sum_{k=0}^3 \lambda_k H_k + \left(\sum_{j=1}^3 \mu_j E_j\right)^\sim, 0, 0, \nu\right) \in \mathfrak{e}_7^C \mid \begin{array}{l} \lambda_k, \nu \in C \\ \mu_j \in C, \mu_1 + \mu_2 + \mu_3 = 0 \end{array} \right\}$$

(where $H_k = -iG_{k4+k}$), then \mathfrak{h} is an abelian subalgebra of \mathfrak{e}_7^C (it will be a Cartan subalgebra of \mathfrak{e}_7^C). In the following calculations, we put $h_\delta = \sum_{k=0}^3 \lambda_k H_k$, $H = \sum_{j=1}^3 \mu_j E_j$.

I The roots of \mathfrak{e}_6^C are also roots of \mathfrak{e}_7^C . Indeed, let α be a root of \mathfrak{e}_6^C and $S \in \mathfrak{e}_6^C \subset \mathfrak{e}_7^C$ be a root vector belonging to α . Then

$$\begin{aligned}
[h, S] &= [\Phi(h_\delta + \tilde{H}, 0, 0, \nu), \Phi(S, 0, 0, 0)] \\
&= \Phi([h_\delta + \tilde{H}, S], 0, 0, 0) = \Phi(\alpha(h_\delta + \tilde{H})S, 0, 0, 0) = \alpha(h)S.
\end{aligned}$$

Hence α is a root of \mathfrak{e}_7^C .

$$\begin{aligned}
\text{II} \quad & [\Phi(h_\delta + \tilde{H}, 0, 0, \nu), \Phi(0, E_j, 0, 0)] \\
&= \Phi\left(0, \left(h_\delta + \tilde{H} + \frac{2}{3}\nu\right)E_j, 0, 0\right) = \left(\mu_j + \frac{2}{3}\nu\right)\Phi(0, E_j, 0, 0), \\
&[\Phi((h_\delta + \tilde{H}, 0, 0, \nu), \Phi(0, 0, E_j, 0))] = \Phi\left(0, 0, \left((h_\delta + \tilde{H})' - \frac{2}{3}\nu\right)E_j, 0\right) \\
&= \Phi\left(0, 0, \left(h_\delta - \tilde{H} - \frac{2}{3}\nu\right)E_j, 0\right) = \left(-\mu_j - \frac{2}{3}\nu\right)\Phi(0, 0, E_j, 0).
\end{aligned}$$

Hence $\pm\left(\mu_j + \frac{2}{3}\nu\right)$, $0 \leq j \leq 3$, are roots of \mathfrak{e}_7^C .

$$\begin{aligned}
\text{III} \quad & [\Phi(h_\delta + \tilde{H}, 0, 0, \nu), \Phi(0, F_1(a), 0, 0)] \quad a = e_k \pm ie_{4+k} \\
&= \Phi\left(0, \left(h_\delta + \tilde{H} + \frac{2}{3}\nu\right)F_1(a), 0, 0\right) \\
&= \Phi\left(0, F_1(h_\delta a) + \frac{1}{2}(\mu_2 + \mu_3)F_1(a) + \frac{2}{3}\nu F_1(a), 0, 0\right)
\end{aligned}$$

$$= \left(\pm \lambda_k - \frac{1}{2}\mu_1 + \frac{2}{3}\nu \right) \Phi(0, F_1(a), 0, 0).$$

Hence $\pm \lambda_k - \frac{1}{2}\mu_1 + \frac{2}{3}\nu$, $0 \leq k \leq 3$, are roots of \mathfrak{e}_7^C . The remainders of roots can be similarly found (νh_δ , $\kappa \pi h_\delta$ in Theorem 3.6.4 will be used).

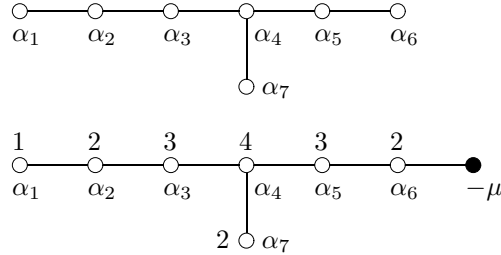
Theorem 4.6.2. *In the root system of Theorem 4.6.1,*

$$\begin{aligned} \alpha_1 &= \lambda_0 - \lambda_1, & \alpha_2 &= \lambda_1 - \lambda_2, & \alpha_3 &= \lambda_2 - \lambda_3, \\ \alpha_4 &= \frac{1}{2}(-\lambda_0 - \lambda_1 - \lambda_2 + \lambda_3) + \frac{1}{2}(\mu_3 - \mu_1), \\ \alpha_5 &= \frac{1}{2}(\lambda_0 + \lambda_1 + \lambda_2 + \lambda_3) + \frac{1}{2}(\mu_1 - \mu_2), \\ \alpha_6 &= \mu_2 + \frac{3}{2}\nu, & \alpha_7 &= -\mu_3 - \frac{3}{2}\nu \end{aligned}$$

is a fundamental root system of the Lie algebra \mathfrak{e}_7^C and

$$\mu = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + 2\alpha_7$$

is the highest root. The Dynkin diagram and the extended Dynkin diagram of \mathfrak{e}_7^C are respectively given by



Proof. In the following, the notation $n_1 n_2 \cdots n_7$ denotes the root $n_1 \alpha_1 + n_2 \alpha_2 + \cdots + n_7 \alpha_7$. Now, all positive roots of \mathfrak{e}_7^C are represented by

$$\begin{array}{ll} \lambda_0 - \lambda_1 = 1 & 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad \lambda_0 + \lambda_1 = 1 \quad 2 \quad 2 \quad 2 \quad 2 \quad 1 \quad 1 \\ \lambda_0 - \lambda_2 = 1 & 1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad \lambda_0 + \lambda_2 = 1 \quad 1 \quad 2 \quad 2 \quad 2 \quad 1 \quad 1 \\ \lambda_0 - \lambda_3 = 1 & 1 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0 \quad \lambda_0 + \lambda_3 = 1 \quad 1 \quad 1 \quad 2 \quad 2 \quad 1 \quad 1 \\ \lambda_1 - \lambda_2 = 0 & 1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad \lambda_1 + \lambda_2 = 0 \quad 1 \quad 2 \quad 2 \quad 2 \quad 1 \quad 1 \\ \lambda_1 - \lambda_3 = 0 & 1 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0 \quad \lambda_1 + \lambda_3 = 0 \quad 1 \quad 1 \quad 2 \quad 2 \quad 1 \quad 1 \\ \lambda_2 - \lambda_3 = 0 & 0 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0 \quad \lambda_2 + \lambda_3 = 0 \quad 0 \quad 1 \quad 2 \quad 2 \quad 1 \quad 1 \\ \\ \lambda_0 + \frac{1}{2}(\mu_2 - \mu_3) = 1 & 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \\ \lambda_1 + \frac{1}{2}(\mu_2 - \mu_3) = 0 & 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \\ \lambda_2 + \frac{1}{2}(\mu_2 - \mu_3) = 0 & 0 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \\ \lambda_3 + \frac{1}{2}(\mu_2 - \mu_3) = 0 & 0 \quad 0 \quad 1 \quad 1 \quad 1 \quad 1 \end{array}$$

$$\begin{aligned}
\lambda_0 - \frac{1}{2}(\mu_2 - \mu_3) &= 1 & 1 & 1 & 1 & 1 & 0 & 0 \\
\lambda_1 - \frac{1}{2}(\mu_2 - \mu_3) &= 0 & 1 & 1 & 1 & 1 & 0 & 0 \\
\lambda_2 - \frac{1}{2}(\mu_2 - \mu_3) &= 0 & 0 & 1 & 1 & 1 & 0 & 0 \\
\lambda_3 - \frac{1}{2}(\mu_2 - \mu_3) &= 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
\frac{1}{2}(-\lambda_0 - \lambda_1 + \lambda_2 - \lambda_3) + \frac{1}{2}(\mu_3 - \mu_1) &= 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
\frac{1}{2}(\lambda_0 + \lambda_1 + \lambda_2 - \lambda_3) + \frac{1}{2}(\mu_3 - \mu_1) &= 1 & 2 & 3 & 3 & 2 & 1 & 1 \\
\frac{1}{2}(-\lambda_0 + \lambda_1 + \lambda_2 + \lambda_3) + \frac{1}{2}(\mu_3 - \mu_1) &= 0 & 1 & 2 & 3 & 2 & 1 & 1 \\
\frac{1}{2}(\lambda_0 - \lambda_1 + \lambda_2 + \lambda_3) + \frac{1}{2}(\mu_3 - \mu_1) &= 1 & 1 & 2 & 3 & 2 & 1 & 1 \\
\frac{1}{2}(\lambda_0 + \lambda_1 - \lambda_2 + \lambda_3) + \frac{1}{2}(\mu_3 - \mu_1) &= 1 & 2 & 2 & 3 & 2 & 1 & 1 \\
\frac{1}{2}(-\lambda_0 - \lambda_1 - \lambda_2 + \lambda_3) + \frac{1}{2}(\mu_3 - \mu_1) &= 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
\frac{1}{2}(\lambda_0 - \lambda_1 - \lambda_2 - \lambda_3) + \frac{1}{2}(\mu_3 - \mu_1) &= 1 & 1 & 1 & 1 & 0 & 0 & 0 \\
\frac{1}{2}(-\lambda_0 + \lambda_1 - \lambda_2 - \lambda_3) + \frac{1}{2}(\mu_3 - \mu_1) &= 0 & 1 & 1 & 1 & 0 & 0 & 0 \\
\frac{1}{2}(\lambda_0 - \lambda_1 + \lambda_2 - \lambda_3) - \frac{1}{2}(\mu_1 - \mu_2) &= 1 & 1 & 2 & 2 & 1 & 1 & 1 \\
\frac{1}{2}(\lambda_0 - \lambda_1 - \lambda_2 + \lambda_3) - \frac{1}{2}(\mu_1 - \mu_2) &= 1 & 1 & 1 & 2 & 1 & 1 & 1 \\
\frac{1}{2}(\lambda_0 + \lambda_1 + \lambda_2 + \lambda_3) + \frac{1}{2}(\mu_1 - \mu_2) &= 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
\frac{1}{2}(\lambda_0 + \lambda_1 - \lambda_2 - \lambda_3) - \frac{1}{2}(\mu_1 - \mu_2) &= 1 & 2 & 2 & 2 & 1 & 1 & 1 \\
\frac{1}{2}(-\lambda_0 + \lambda_1 - \lambda_2 + \lambda_3) - \frac{1}{2}(\mu_1 - \mu_2) &= 0 & 1 & 1 & 2 & 1 & 1 & 1 \\
\frac{1}{2}(-\lambda_0 + \lambda_1 + \lambda_2 - \lambda_3) - \frac{1}{2}(\mu_1 - \mu_2) &= 0 & 1 & 2 & 2 & 1 & 1 & 1 \\
\frac{1}{2}(\lambda_0 + \lambda_1 + \lambda_2 + \lambda_3) - \frac{1}{2}(\mu_1 - \mu_2) &= 1 & 2 & 3 & 4 & 3 & 2 & 2 \\
\frac{1}{2}(-\lambda_0 - \lambda_1 + \lambda_2 + \lambda_3) - \frac{1}{2}(\mu_1 - \mu_2) &= 0 & 0 & 1 & 2 & 1 & 1 & 1 \\
-\mu_1 - \frac{2}{3}\nu &= 1 & 2 & 3 & 4 & 2 & 1 & 2 \\
\mu_2 + \frac{2}{3}\nu &= 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
-\mu_3 - \frac{2}{3}\nu &= 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{aligned}$$

$$\begin{aligned}
\lambda_0 - \frac{1}{2}\mu_1 + \frac{2}{3}\nu &= 1 & 1 & 1 & 1 & 1 & 1 & 0 \\
\lambda_1 - \frac{1}{2}\mu_1 + \frac{2}{3}\nu &= 0 & 1 & 1 & 1 & 1 & 1 & 0 \\
\lambda_2 - \frac{1}{2}\mu_1 + \frac{2}{3}\nu &= 0 & 0 & 1 & 1 & 1 & 1 & 0 \\
\lambda_3 - \frac{1}{2}\mu_1 + \frac{2}{3}\nu &= 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
\lambda_0 + \frac{1}{2}\mu_1 - \frac{2}{3}\nu &= 1 & 1 & 1 & 1 & 1 & 0 & 1 \\
\lambda_1 + \frac{1}{2}\mu_1 - \frac{2}{3}\nu &= 0 & 1 & 1 & 1 & 1 & 0 & 1 \\
\lambda_2 + \frac{1}{2}\mu_1 - \frac{2}{3}\nu &= 0 & 0 & 1 & 1 & 1 & 0 & 1 \\
\lambda_3 + \frac{1}{2}\mu_1 - \frac{2}{3}\nu &= 0 & 0 & 0 & 1 & 1 & 0 & 1 \\
\frac{1}{2}(-\lambda_0 - \lambda_1 + \lambda_2 - \lambda_3) + \frac{1}{2}\mu_2 - \frac{2}{3}\nu &= 0 & 0 & 1 & 1 & 0 & 0 & 1 \\
\frac{1}{2}(\lambda_0 + \lambda_1 + \lambda_2 - \lambda_3) + \frac{1}{2}\mu_2 - \frac{2}{3}\nu &= 1 & 2 & 3 & 3 & 2 & 1 & 2 \\
\frac{1}{2}(-\lambda_0 + \lambda_1 + \lambda_2 + \lambda_3) + \frac{1}{2}\mu_2 - \frac{2}{3}\nu &= 0 & 1 & 2 & 3 & 2 & 1 & 2 \\
\frac{1}{2}(\lambda_0 - \lambda_1 + \lambda_2 + \lambda_3) + \frac{1}{2}\mu_2 - \frac{2}{3}\nu &= 1 & 1 & 2 & 3 & 2 & 1 & 2 \\
\frac{1}{2}(\lambda_0 + \lambda_1 - \lambda_2 + \lambda_3) + \frac{1}{2}\mu_2 - \frac{2}{3}\nu &= 1 & 2 & 2 & 3 & 2 & 1 & 2 \\
\frac{1}{2}(-\lambda_0 - \lambda_1 - \lambda_2 + \lambda_3) + \frac{1}{2}\mu_2 - \frac{2}{3}\nu &= 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
\frac{1}{2}(\lambda_0 - \lambda_1 - \lambda_2 - \lambda_3) + \frac{1}{2}\mu_2 - \frac{2}{3}\nu &= 1 & 1 & 1 & 1 & 0 & 0 & 1 \\
\frac{1}{2}(-\lambda_0 + \lambda_1 - \lambda_2 - \lambda_3) + \frac{1}{2}\mu_2 - \frac{2}{3}\nu &= 0 & 1 & 1 & 1 & 0 & 0 & 1 \\
\frac{1}{2}(\lambda_0 - \lambda_1 + \lambda_2 - \lambda_3) + \frac{1}{2}\mu_3 - \frac{2}{3}\nu &= 1 & 1 & 2 & 2 & 1 & 0 & 1 \\
\frac{1}{2}(\lambda_0 - \lambda_1 - \lambda_2 + \lambda_3) + \frac{1}{2}\mu_3 + \frac{2}{3}\nu &= 1 & 1 & 1 & 2 & 1 & 0 & 1 \\
\frac{1}{2}(\lambda_0 + \lambda_1 + \lambda_2 + \lambda_3) - \frac{1}{2}\mu_3 - \frac{2}{3}\nu &= 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
\frac{1}{2}(\lambda_0 + \lambda_1 - \lambda_2 - \lambda_3) + \frac{1}{2}\mu_3 - \frac{2}{3}\nu &= 1 & 2 & 2 & 2 & 1 & 0 & 1 \\
\frac{1}{2}(-\lambda_0 + \lambda_1 - \lambda_2 + \lambda_3) + \frac{1}{2}\mu_3 - \frac{2}{3}\nu &= 0 & 1 & 1 & 2 & 1 & 0 & 1 \\
\frac{1}{2}(-\lambda_0 + \lambda_1 + \lambda_2 - \lambda_3) + \frac{1}{2}\mu_3 - \frac{2}{3}\nu &= 0 & 1 & 2 & 2 & 1 & 0 & 1 \\
\frac{1}{2}(\lambda_0 + \lambda_1 + \lambda_2 + \lambda_3) + \frac{1}{2}\mu_3 - \frac{2}{3}\nu &= 1 & 2 & 3 & 4 & 3 & 1 & 2 \\
\frac{1}{2}(-\lambda_0 - \lambda_1 + \lambda_2 + \lambda_3) + \frac{1}{2}\mu_3 - \frac{2}{3}\nu &= 0 & 0 & 1 & 2 & 1 & 0 & 1.
\end{aligned}$$

Hence $\Pi = \{\alpha_1, \alpha_2, \dots, \alpha_7\}$ is a fundamental root system of \mathfrak{e}_7^C . The real part $\mathfrak{h}_{\mathbf{R}}$ of \mathfrak{h} is

$$\mathfrak{h}_{\mathbf{R}} = \left\{ \Phi \left(\sum_{k=0}^3 \lambda_k H_k + \left(\sum_{j=1}^3 \mu_j E_j \right)^{\sim}, 0, 0, \nu \right) \in \mathfrak{e}_7^C \mid \begin{array}{l} \lambda_k, \nu \in \mathbf{R} \\ \mu_j \in \mathbf{R}, \mu_1 + \mu_2 + \mu_3 = 0 \end{array} \right\}$$

and the Killing form B_7 of \mathfrak{e}_7^C on $\mathfrak{h}_{\mathbf{R}}$ is given by

$$B_7(h, h') = 6 \left(6 \sum_{k=0}^3 \lambda_k \lambda_k' + 3 \sum_{j=1}^3 \mu_j \mu_j' + 4\nu\nu' \right)$$

for $h = \Phi \left(\sum_{k=0}^3 \lambda_k H_k + \left(\sum_{j=1}^3 \mu_j E_j \right)^{\sim}, 0, 0, \nu \right)$, $h' = \Phi \left(\sum_{k=0}^3 \lambda_k' H_k + \left(\sum_{j=1}^3 \mu_j' E_j \right)^{\sim}, 0, 0, \nu' \right) \in \mathfrak{h}_{\mathbf{R}}$. Indeed, from Theorem 4.5.2, we have

$$\begin{aligned} B_7(h, h') &= \frac{3}{2} B_6 \left(\sum_{k=0}^3 \lambda_k H_k + \left(\sum_{j=1}^3 \mu_j E_j \right)^{\sim}, \sum_{k=0}^3 \lambda_k' H_k + \left(\sum_{j=1}^3 \mu_j' E_j \right)^{\sim} \right) + 24\nu\nu' \\ &= \frac{3}{2} 12 \left(2 \sum_{k=0}^3 \lambda_k \lambda_k' + \sum_{j=1}^3 \mu_j \mu_j' \right) + 24\nu\nu' \text{ (Theorem 3.6.5)} \\ &= 6 \left(6 \sum_{k=0}^3 \lambda_k \lambda_k' + 3 \sum_{j=1}^3 \mu_j \mu_j' + 4\nu\nu' \right). \end{aligned}$$

Now, the canonical elements $H_{\alpha_i} \in \mathfrak{h}_{\mathbf{R}}$ corresponding to α_i ($B_7(H_{\alpha_i}, H) = \alpha_i(H)$, $H \in \mathfrak{h}$) are determined as follows.

$$\begin{aligned} H_{\alpha_1} &= \frac{1}{36} \Phi(H_0 - H_1, 0, 0, 0), \\ H_{\alpha_2} &= \frac{1}{36} \Phi(H_1 - H_2, 0, 0, 0), \\ H_{\alpha_3} &= \frac{1}{36} \Phi(H_2 - H_3, 0, 0, 0), \\ H_{\alpha_4} &= \frac{1}{72} \Phi((-H_0 - H_1 - H_2 + H_3) + 2(E_3 - E_1)^{\sim}, 0, 0, 0), \\ H_{\alpha_5} &= \frac{1}{72} \Phi((H_0 + H_1 + H_2 + H_3) + 2(E_1 - E_2)^{\sim}, 0, 0, 0), \\ H_{\alpha_6} &= \frac{1}{54} \Phi((-E_1 + 2E_2 - E_3)^{\sim}, 0, 0, \frac{3}{2}), \\ H_{\alpha_7} &= \frac{1}{54} \Phi((E_1 + E_2 - 2E_3)^{\sim}, 0, 0, -\frac{3}{2}). \end{aligned}$$

Thus we have

$$(\alpha_1, \alpha_1) = B_7(H_{\alpha_1}, H_{\alpha_1}) = 36 \frac{1}{36} \frac{1}{36} 2 = \frac{1}{18}$$

and the other inner products are similarly calculated. Consequently, the inner product induced by the Killing form B_7 between $\alpha_1, \alpha_2, \dots, \alpha_7$ and $-\mu$ are given by

$$\begin{aligned}
(\alpha_i, \alpha_i) &= \frac{1}{18}, \quad i = 1, 2, 3, 4, 5, 6, 7, \\
(\alpha_1, \alpha_2) &= (\alpha_2, \alpha_3) = (\alpha_3, \alpha_4) = (\alpha_4, \alpha_5) = (\alpha_4, \alpha_7) = (\alpha_5, \alpha_6) = -\frac{1}{36}, \\
(\alpha_i, \alpha_j) &= 0, \quad \text{otherwise}, \\
(-\mu, -\mu) &= \frac{1}{18}, \quad (-\mu, \alpha_6) = -\frac{1}{36}, \quad (-\mu, \alpha_i) = 0, \quad i = 1, 2, 3, 5, 7,
\end{aligned}$$

using them, we can draw the Dynkin diagram and the extended Dynkin diagram of \mathfrak{e}_7^C .

According to Borel-Siebenthal theory, the Lie algebra \mathfrak{e}_7 has four subalgebras as maximal subalgebras with the maximal rank 7.

(1) The first one is a subalgebra of type $T \oplus E_6$ which is obtained as the fixed points of an involution ι of \mathfrak{e}_7 .

(2) The second one is a subalgebra of type $A_1 \oplus D_6$ which is obtained as the fixed points of an involution σ of \mathfrak{e}_7 .

(3) The third one is a subalgebra of type A_7 which is obtained as the fixed points of an involution $\lambda\gamma$ of \mathfrak{e}_7 .

(4) The fourth one is a subalgebra of type $A_2 \oplus A_5$ which is obtained as the fixed points of an automorphism w of order 3 of \mathfrak{e}_7 .

These subalgebras will be realized as subgroups of the group E_7 in Theorems 4.10.2, 4.11.15, 4.12.5 and 4.13.5, respectively.

4.7. Subgroups E_6 and $U(1)$ of E_7

We shall study the following subgroup $(E_7)_{(0,0,1,0)}$ of E_7 :

$$(E_7)_{(0,0,1,0)} = \{\alpha \in E_7 \mid \alpha(0, 0, 1, 0) = (0, 0, 1, 0)\}.$$

Lemma 4.7.1. *If $\alpha \in E_7$ satisfies $\alpha(0, 0, 1, 0) = (0, 0, 1, 0)$, then α also satisfies $\alpha(0, 0, 0, 1) = (0, 0, 0, 1)$, and conversely.*

Proof. If $\alpha \in E_7$ satisfies $\alpha(0, 0, 1, 0) = (0, 0, 1, 0)$, then we have

$$\alpha(0, 0, 0, 1) = \alpha(-\tau\lambda(0, 0, 1, 0)) = -\tau\lambda\alpha(0, 0, 1, 0) = -\tau\lambda(0, 0, 1, 0) = (0, 0, 0, 1).$$

The converse can be similarly proved.

Theorem 4.7.2. $(E_7)_{(0,0,1,0)} \cong E_6$.

Proof. We associate an element $\alpha \in E_6 = \{\alpha \in \text{Iso}_C(\mathfrak{J}^C) \mid \det(\alpha X) = \det X, \langle \alpha X, \alpha Y \rangle = \langle X, Y \rangle\}$ with the element

$$\tilde{\alpha} = \begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & \tau\alpha\tau & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in (E_7)_{(0,0,1,0)} \subset E_7.$$

We first have to prove that $\tilde{\alpha} \in E_7$. For $P = (X, Y, \xi, \eta)$, $Q = (Z, W, \zeta, \omega) \in \mathfrak{P}^C$, we have

$$\begin{aligned} \tilde{\alpha}P \times \tilde{\alpha}Q &= (\alpha X, \tau\alpha\tau Y, \xi, \eta) \times (\alpha Z, \tau\alpha\tau W, \zeta, \omega) \\ &= \dots (\text{using } \alpha(X \vee Y)\alpha^{-1} = \alpha X \vee \tau\alpha\tau Y, (\alpha\phi\alpha^{-1})' = (\tau\alpha\tau)\phi'(\tau\alpha^{-1}\tau) \text{ etc.}) \dots \\ &= \tilde{\alpha}(P \times Q)\tilde{\alpha}^{-1} \end{aligned}$$

and $\langle \tilde{\alpha}P, \tilde{\alpha}Q \rangle = \langle P, Q \rangle$ is evident. Hence $\tilde{\alpha} \in E_7$, moreover $\tilde{\alpha} \in (E_7)_{(0,0,1,0)}$.

Conversely, suppose that $\alpha \in E_7$ satisfies $\alpha(0, 0, 1, 0) = (0, 0, 1, 0)$ and $\alpha(0, 0, 0, 1) = (0, 0, 0, 1)$ (Lemma 4.7.1). Then α is of the form

$$\alpha = \begin{pmatrix} \beta & \epsilon & 0 & 0 \\ \delta & \beta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \beta, \beta_1, \delta, \epsilon \in \text{Hom}_C(\mathfrak{J}^C).$$

Indeed, the fact that the left bottom parts are 0 follows from

$$\begin{aligned} \langle \alpha \dot{X}, \dot{1} \rangle &= \langle \alpha \dot{X}, \alpha \dot{1} \rangle = \langle \dot{X}, \dot{1} \rangle = 0, \\ \langle \alpha \dot{X}, \dot{1} \rangle &= \langle \alpha \dot{X}, \alpha \dot{1} \rangle = \langle \dot{X}, \dot{1} \rangle = 0. \end{aligned}$$

Now, since

$$\mathfrak{M}^C \ni \alpha \begin{pmatrix} X \\ \frac{1}{\eta}X \times X \\ \frac{1}{\eta^2}\det X \\ \eta \end{pmatrix} = \begin{pmatrix} \beta X + \frac{1}{\eta}\epsilon(X \times X) \\ \delta X + \frac{1}{\eta}\beta_1(X \times X) \\ \frac{1}{\eta^2}\det X \\ \eta \end{pmatrix},$$

we can see that

$$\left(\beta X + \frac{1}{\eta}\epsilon(X \times X) \right) \times \left(\beta X + \frac{1}{\eta}\epsilon(X \times X) \right) = \eta \left(\delta X + \frac{1}{\eta}\beta_1(X \times X) \right)$$

holds for all $0 \neq \eta \in C$. Comparing the coefficients of η of both sides, we have $\delta = 0$.

Similarly, from $\alpha \left(\frac{1}{\xi}(Y \times Y), Y, \xi, \frac{1}{\xi^2}\det Y \right) \in \mathfrak{M}^C$, we have $\epsilon = 0$. Furthermore, since

$$\mathfrak{M}^C \ni \alpha(X, X \times X, \det X, 1) = (\beta X, \beta_1(X \times X), \det X, 1),$$

we have

$$\beta X \times \beta X = \beta_1(X \times X), \quad (\beta X, \beta_1(X \times X)) = 3\det X.$$

and so

$$\det(\beta X) = \frac{1}{3}(\beta X, \beta X \times \beta X) = \frac{1}{3}(\beta X, \beta_1(X \times X)) = \det X,$$

which implies that $\beta \in E_6^C$. The equality $\langle \alpha \dot{X}, \alpha \dot{Y} \rangle = \langle \dot{X}, \dot{Y} \rangle$ implies $\langle \beta X, \beta Y \rangle = \langle X, Y \rangle$ and therefore $\beta \in E_6$. Moreover from the relation

$$\beta_1(X \times X) = \beta X \times \beta X = \tau \beta \tau(X \times X),$$

we obtain $\beta_1 = \tau \beta \tau$. Indeed, putting $X \times X$ instead of X , we have

$$(\det X) \beta_1 X = (\det X) \tau \beta \tau X.$$

If $\det X \neq 0$, then we have $\beta_1 X = \tau \beta \tau X$. Since β_1 and $\tau \beta \tau$ are linear mappings (of course are continuous mappings), we have $\beta_1 X = \tau \beta \tau X$ even if $\det X = 0$. Thus, the proof of Theorem 4.7.2 is completed.

For $\theta \in C, \theta \neq 0$, we define a C -linear transformation $\varphi_1(\theta) : \mathfrak{P}^C \rightarrow \mathfrak{P}^C$ by

$$\varphi_1(\theta)(X, Y, \xi, \eta) = (\theta^{-1}X, \theta Y, \theta^3 \xi, \theta^{-3} \eta).$$

Then $\varphi_1(\theta) \in E_7^C$.

Theorem 4.7.3. *The group E_7 contains*

$$U(1) = \{\varphi_1(\theta) \mid \theta \in C, (\tau \theta) \theta = 1\}$$

as a subgroup. This subgroup is isomorphic to the usual unitary group $U(1) = \{\theta \in C \mid (\tau \theta) \theta = 1\}$.

Proof. It is easy to check that $\varphi_1(\theta) \in E_7$.

4.8. Connectedness of E_7

We denote by $(E_7)_0$ the connected component of E_7 containing the identity 1.

Lemma 4.8.1. *For $a \in C$, if we define a mapping $\alpha_i(a) : \mathfrak{P}^C \rightarrow \mathfrak{P}^C$, $i = 1, 2, 3$ by*

$$\alpha_i(a) = \begin{pmatrix} 1 + (\cos |a| - 1)p_i & 2a \frac{\sin |a|}{|a|} E_i & 0 & -\tau a \frac{\sin |a|}{|a|} E_i \\ -2\tau a \frac{\sin |a|}{|a|} E_i & 1 + (\cos |a| - 1)p_i & a \frac{\sin |a|}{|a|} E_i & 0 \\ 0 & -\tau a \frac{\sin |a|}{|a|} E_i & \cos |a| & 0 \\ a \frac{\sin |a|}{|a|} E_i & 0 & 0 & \cos |a| \end{pmatrix}$$

(if $a = 0$, then $\frac{\sin |a|}{|a|}$ means 1), then $\alpha_i(a) \in (E_7)_0$, where $p_i : \mathfrak{J}^C \rightarrow \mathfrak{J}^C$ is the C -linear mapping defined by

$$p_i \begin{pmatrix} \xi_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & \xi_2 & x_1 \\ x_2 & \bar{x}_1 & \xi_3 \end{pmatrix} = \begin{pmatrix} \xi_1 & \delta_{i3}x_3 & \delta_{i2}\bar{x}_2 \\ \delta_{i3}\bar{x}_3 & \xi_2 & \delta_{i1}x_1 \\ \delta_{i2}x_2 & \delta_{i1}\bar{x}_1 & \xi_3 \end{pmatrix},$$

where δ_{ij} is the Kronecker delta symble. The mappings $\alpha_1(a_1), \alpha_2(a_2), \alpha(a_3), (a_i \in C)$ are commutative for each other.

Proof. For

$$\Phi_i(a) = \Phi(0, -\tau a E_i, \tau a E_i, 0) = \begin{pmatrix} 0 & 2aE_i & 0 & -\tau a E_i \\ -2\tau a E_i & 0 & aE_i & 0 \\ 0 & -\tau a E_i & 0 & 0 \\ aE_i & 0 & 0 & 0 \end{pmatrix} \in \mathfrak{e}_7$$

(Theorem 4.3.4), we have $\alpha_i(a) = \exp \Phi_i(a)$. Hence $\alpha_i(a) \in (E_7)_0$. The relation $[\Phi_i(a_i), \Phi_j(a_j)] = 0$ shows that $\alpha_i(a_i)$ and $\alpha_j(a_j)$ are commutative.

Proposition 4.8.2. Any element $P \in \mathfrak{M}^C$, $P \neq 0$ can be transformed to a diagonal form by some element $\alpha \in (E_7)_0$:

$$\alpha P = (X, Y, \xi, \eta), \quad X, Y \text{ are diagonal, } \xi > 0.$$

Proof. Let $P = (X, Y, \xi, \eta) \in \mathfrak{M}^C$. We shall first show that P can be transformed to a diagonal form with $\xi \neq 0$.

(1) Case $P = (X, Y, \xi, \eta)$, $\xi \neq 0$. In this case, $X = \frac{1}{\xi}(Y \times Y)$. Choose $\beta \in E_6$ such that $\tau\beta\tau Y$ is diagonal (Proposition 3.8.2), then

$$\beta X = \frac{1}{\xi}\beta(Y \times Y) = \frac{1}{\xi}\tau\beta\tau Y \times \tau\beta\tau Y$$

is also diagonal.

(2) Case $P = (X, Y, 0, \eta)$, $Y \neq 0$. Choose $\beta \in E_6$ so that

$$\tau\beta\tau Y = \begin{pmatrix} \eta_1 & 0 & 0 \\ 0 & \eta_2 & 0 \\ 0 & 0 & \eta_3 \end{pmatrix}, \quad \eta_i \in C$$

(Proposition 3.8.2). Since $\tau\beta\tau Y \neq 0$, some η_i is non-zero: $\eta_i \neq 0$. Applying $\alpha_i(-\pi/2) \in (E_7)_0$ of Lemma 4.8.1 on βP , we get

$$\alpha_i(-\pi/2)\beta P = \begin{pmatrix} 1-p_i & -2E_i & 0 & E_i \\ 2E_i & 1-p_i & -E_i & 0 \\ 0 & E_i & 0 & 0 \\ -E_i & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \beta X \\ \tau\beta\tau Y \\ 0 \\ \eta \end{pmatrix} = \begin{pmatrix} * \\ * \\ \eta_i \\ * \end{pmatrix}, \quad \eta_i \neq 0,$$

so that this case is reduced to the case (1).

(3) Case $P = (X, 0, 0, \eta)$, $X \neq 0$. Choose $\beta \in E_6$ so that $\beta X = \xi_1 E_1 + \xi_2 E_2 + \xi_3 E_3$, $\xi_i \in C$ (Proposition 3.8.2). Since $\beta X \neq 0$, some ξ_i is non-zero: $\xi_i \neq 0$. Then

$$\alpha_{i+1}(-\pi/2)\beta P = (*, \xi_i E_{i+2} + \xi_{i+2} E_i, 0, *), \quad \xi_i \neq 0,$$

so that this case is reduced to the case (2).

(4) Case $P = (0, 0, 0, \eta)$, $\eta \neq 0$. Then

$$\alpha_1(-\pi/2)P = (\eta E_1, 0, 0, 0), \quad \eta \neq 0,$$

so that this case is also reduced to the case (3).

Consequently, any element P can be transformed to a diagonal form with $\xi \neq 0$. Furthermore, by applying some $\phi_1(\theta) \in U(1) \subset (E_7)_0$ of Theorem 4.7.3 on it, then ξ becomes $\xi > 0$. Thus the proof of Proposition 4.8.2 is completed.

Remark. In Proposition 4.8.2, the condition $P \in \mathfrak{M}^C$ does not need. That is, any element $P \in \mathfrak{P}^C$ can be transformed to a diagonal form by some $\alpha \in E_7$. (See Miyasaka, Yasukura and Yokota [23]).

We define a space \mathfrak{M}_1 , called the compact Freudenthal manifold, by

$$\mathfrak{M}_1 = \{P \in \mathfrak{P}^C \mid P \times P = 0, \langle P, P \rangle = 1\}.$$

Theorem 4.8.3. $E_7/E_6 \simeq \mathfrak{M}_1$.

In particular, the group E_7 is connected.

Proof. For $\alpha \in E_7$ and $P \in \mathfrak{M}_1$, we have $\alpha P \in \mathfrak{M}_1$. Hence E_7 acts on \mathfrak{M}_1 . We shall prove that the group $(E_7)_0$ acts transitively on \mathfrak{M}_1 . To prove this, it is sufficient to show that any element $P \in \mathfrak{M}_1$ can be transformed to $(0, 0, 1, 0) \in \mathfrak{M}_1$ by some $\alpha \in (E_7)_0$. Now, $P \in \mathfrak{M}_1$ can be transformed to a diagonal form by $\alpha \in (E_7)_0$:

$$\alpha P = \left(\frac{1}{\xi} \begin{pmatrix} \eta_2 \eta_3 & 0 & 0 \\ 0 & \eta_3 \eta_1 & 0 \\ 0 & 0 & \eta_1 \eta_2 \end{pmatrix}, \begin{pmatrix} \eta_1 & 0 & 0 \\ 0 & \eta_2 & 0 \\ 0 & 0 & \eta_3 \end{pmatrix}, \xi, \frac{1}{\xi^2} \eta_1 \eta_2 \eta_3 \right), \quad \xi > 0$$

(Proposition 4.8.2). From the condition $\langle \alpha P, \alpha P \rangle = \langle P, P \rangle = 1$, we have

$$\frac{1}{\xi^2} (|\eta_2 \eta_3|^2 + |\eta_3 \eta_1|^2 + |\eta_1 \eta_2|^2) + (|\eta_1|^2 + |\eta_2|^2 + |\eta_3|^2) + \xi^2 + \frac{1}{\xi^4} |\eta_1 \eta_2 \eta_3|^2 = 1,$$

that is,

$$\left(1 + \frac{|\eta_1|^2}{\xi^2}\right) \left(1 + \frac{|\eta_2|^2}{\xi^2}\right) \left(1 + \frac{|\eta_3|^2}{\xi^2}\right) = \frac{1}{\xi^2}. \quad (\text{i})$$

Choose $r_1, r_2, r_3 \in \mathbf{R}$, $0 \leq r_i < \frac{\pi}{2}$, such that

$$\tan r_i = \frac{|\eta_i|}{\xi}, \quad i = 1, 2, 3,$$

then (i) becomes

$$\xi = \cos r_1 \cos r_2 \cos r_3.$$

By letting

$$a_i = \frac{\eta_i}{|\eta_i|} r_i, \quad i = 1, 2, 3$$

(if $\eta_i = 0$, then a_i means 0), we have

$$r_i = |a_i|, \quad \eta_i = \frac{1}{|a_i|} \frac{\eta_i}{|\eta_i|} r_i \frac{|\eta_i|}{\xi} \xi = \frac{a_i}{|a_i|} \tan r_i \cos r_1 \cos r_2 \cos r_3.$$

Therefore, we see that αP is of the form

$$\begin{pmatrix} \begin{pmatrix} \cos |a_1| a_2 \frac{\sin |a_2|}{|a_2|} a_3 \frac{\sin |a_3|}{|a_3|} & 0 & 0 \\ 0 & a_1 \frac{\sin |a_1|}{|a_1|} \cos |a_2| a_3 \frac{\sin |a_3|}{|a_3|} & 0 \\ 0 & 0 & a_1 \frac{\sin |a_1|}{|a_1|} a_2 \frac{\sin |a_2|}{|a_2|} \cos |a_3| \end{pmatrix} \\ \begin{pmatrix} a_1 \frac{\sin |a_1|}{|a_1|} \cos |a_2| \cos |a_3| & 0 & 0 \\ 0 & \cos |a_1| a_2 \frac{\sin |a_2|}{|a_2|} \cos |a_3| & 0 \\ 0 & 0 & \cos |a_1| \cos |a_2| a_3 \frac{\sin |a_3|}{|a_3|} \end{pmatrix} \\ \cos |a_1| \cos |a_2| \cos |a_3| \\ a_1 \frac{\sin |a_1|}{|a_1|} a_2 \frac{\sin |a_2|}{|a_2|} a_3 \frac{\sin |a_3|}{|a_3|} \end{pmatrix}$$

$$= \alpha_3(a_3) \alpha_2(a_2) \alpha_1(a_1) (0, 0, 1, 0),$$

hence we have

$$\alpha_1(a_1)^{-1} \alpha_2(a_2)^{-1} \alpha_3(a_3)^{-1} \alpha P = (0, 0, 1, 0).$$

This shows the transitivity of $(E_7)_0$. Since we have $\mathfrak{M}_1 = (E_7)_0(0, 0, 1, 0)$, \mathfrak{M}_1 is connected. Now, the group E_7 acts transitively on \mathfrak{M}_1 and the isotropy subgroup of E_7 at $(0, 0, 1, 0) \in \mathfrak{M}_1$ is E_6 (Theorem 4.7.2). Therefore we have the homeomorphism $E_7/E_6 \simeq \mathfrak{M}_1$. Finally, the connectedness of E_7 follows from the connectedness of \mathfrak{M}_1 and E_6 .

4.9. Center $z(E_7)$ of E_7

Theorem 4.9.1. *The center $z(E_7)$ of the group E_7 is the cyclic group of order 2:*

$$z(E_7) = \{1, -1\}.$$

Proof. Let $\alpha \in z(E_7)$. From the commutativity with $\beta \in E_6 \subset E_7$, we have $\beta \alpha(0, 0, 1, 0) = \alpha \beta(0, 0, 1, 0) = \alpha(0, 0, 1, 0)$. If we denote $\alpha(0, 0, 1, 0) = (X, Y, \xi, \eta) \in \mathfrak{P}^C$, then from $(\beta X, \tau \beta \tau Y, \xi, \eta) = (X, Y, \xi, \eta)$, we have

$$\beta X = X, \quad \tau \beta \tau Y = Y \quad \text{for all } \beta \in E_6.$$

Hence $X = Y = 0$. Therefore, $\alpha(0, 0, 1, 0)$ is of the form

$$\alpha(0, 0, 1, 0) = (0, 0, \xi, \eta).$$

From the condition $\alpha(0, 0, 1, 0) \in \mathfrak{M}^C$, we have $\xi\eta = 0$. Suppose $\xi = 0$, then $\alpha(0, 0, 1, 0) = (0, 0, 0, \eta)$, $\eta \neq 0$. Also from the commutativity with $\varphi_1(\theta) \in U(1) \subset E_7$ (Theorem 4.7.3), we have

$$\begin{aligned} (0, 0, 0, \theta^{-3}\eta) &= \varphi_1(\theta)(0, 0, 0, \eta) = \varphi_1(\theta)\alpha(0, 0, 1, 0) \\ &= \alpha\varphi_1(\theta)(0, 0, 1, 0) = \alpha(0, 0, \theta^3, 0) = (0, 0, 0, \theta^3\eta), \end{aligned}$$

and so $\theta^{-3}\eta = \theta^3\eta$ for all θ . But this is a contradiction. Hence $\xi \neq 0$, $\eta = 0$, that is, $\alpha(0, 0, 1, 0) = (0, 0, \xi, 0)$. Similarly we have $\alpha(0, 0, 0, 1) = (0, 0, 0, \zeta)$. Since $\{\alpha(0, 0, 1, 0), \alpha(0, 0, 0, 1)\} = 1$, we have $\xi\zeta = 1$, and therefore

$$\alpha(0, 0, 1, 0) = (0, 0, \xi, 0), \quad \alpha(0, 0, 0, 1) = (0, 0, 0, \xi^{-1}).$$

Moreover, from the commutativity with $\lambda \in E_7$,

$$\begin{aligned} (0, 0, 0, -\xi) &= \lambda(0, 0, \xi, 0) = \lambda\alpha(0, 0, 1, 0) \\ &= \alpha\lambda(0, 0, 1, 0) = \alpha(0, 0, 0, -1) = (0, 0, 0, -\xi^{-1}). \end{aligned}$$

Hence $\xi = \xi^{-1}$, that is, $\xi = \pm 1$. In the case $\xi = 1$, we have $\alpha \in E_6$ (Theorem 4.7.2), so that $\alpha \in z(E_6) = \{1, \omega 1, \omega^2 1\}$ (Theorem 3.9.1), that is,

$$\alpha = \begin{pmatrix} \omega' 1 & 0 & 0 & 0 \\ 0 & \omega'^{-1} 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \omega' = 1, \omega \text{ or } \omega^2.$$

Again from the commutativity with λ ,

$$\begin{aligned} (0, \omega' X, 0, 0) &= -\lambda(\omega' X, 0, 0, 0) = -\lambda\alpha(X, 0, 0, 0) \\ &= -\alpha\lambda(X, 0, 0, 0) = \alpha(0, X, 0, 0) = (0, \omega'^{-1} X, 0, 0), \end{aligned}$$

for all $X \in \mathfrak{J}^C$, which shows that $\omega' = \omega'^{-1}$, hence $\omega' = 1$. Therefore $\alpha = 1$. In the case $\xi = -1$, we have $-\alpha \in z(E_6)$, so that by the similar argument as above we have $-\alpha = 1$. Thus we have $z(E_7) = \{1, -1\}$.

According to a general theory of compact Lie groups, it is known that the center of the simply connected compact simple Lie group of type E_7 is the cyclic group of order 2. Hence the group E_7 has to be simply connected. Thus we have the following theorem.

Theorem 4.9.2. $E_7 = \{\alpha \in \text{Iso}_C(\mathfrak{P}^C) \mid \alpha(P \times Q)\alpha^{-1} = \alpha P \times \alpha Q, \langle \alpha P, \alpha Q \rangle = \langle P, Q \rangle\}$ is a simply connected compact Lie group of type E_7 .

4.10. Involution ι and subgroup $(U(1) \times E_6)/\mathbf{Z}_3$ of E_7

Definition. We define a C -linear transformation ι of \mathfrak{P}^C by

$$\iota(X, Y, \xi, \eta) = (-iX, iY, -i\xi, i\eta).$$

Then, $\iota = \varphi_1(i) \in U(1) \subset E_7$, $\iota^2 = -1 \in z(E_7)$ (Theorem 4.9.1) and so $\iota^4 = 1$.

Lemma 4.10.1. ι is conjugate to λ in E_7 .

Proof. For $\delta = \alpha_1\left(\frac{i\pi}{4}\right)\alpha_2\left(\frac{i\pi}{4}\right)\alpha_3\left(\frac{i\pi}{4}\right)$ (Lemma 4.8.1), we have $\iota = \delta^{-1}\lambda\delta$.

ι induces an involutive automorphism $\tilde{\iota} : E_7 \rightarrow E_7$ by

$$\tilde{\iota}(\alpha) = \iota\alpha\iota^{-1}, \quad \alpha \in E_7.$$

We shall now study the following subgroup $(E_7)^\iota$ of E_7 :

$$\begin{aligned} (E_7)^\iota &= \{\alpha \in E_7 \mid \iota\alpha = \alpha\iota\} \\ &\cong \{\alpha \in E_7 \mid \lambda\alpha = \alpha\lambda\} = (E_7)^\lambda. \end{aligned}$$

Theorem 4.10.2 $(E_7)^\iota \cong (U(1) \times E_6)/\mathbf{Z}_3$, $\mathbf{Z}_3 = \{(1, 1), (\omega, \omega 1), (\omega^2, \omega^2 1)\}$,
 $\omega = -\frac{1}{2} + \frac{\sqrt{3}}{2}i \in C$.

Proof. We define a mapping $\varphi : U(1) \times E_6 \rightarrow (E_7)^\iota$ by

$$\varphi(\theta, \beta) = \varphi_1(\theta)\beta, \quad \varphi_1(\theta) = \begin{pmatrix} \theta^{-1}1 & 0 & 0 & 0 \\ 0 & \theta 1 & 0 & 0 \\ 0 & 0 & \theta^3 & 0 \\ 0 & 0 & 0 & \theta^{-3} \end{pmatrix}, \quad \beta = \begin{pmatrix} \beta & 0 & 0 & 0 \\ 0 & \tau\beta\tau & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Evidently $\varphi(\theta, \beta) \in (E_7)^\iota$. Since $\varphi_1(\theta)$ and β are commutative, φ is a homomorphism. We shall prove that φ is onto. Let $\alpha \in (E_7)^\iota$. From $\iota\alpha = \alpha\iota$, α is seen to be of the form

$$\alpha = \begin{pmatrix} \beta & 0 & M & 0 \\ 0 & \delta & 0 & N \\ a & 0 & \mu & 0 \\ 0 & b & 0 & \nu \end{pmatrix}, \quad \begin{aligned} &\beta, \delta \in \text{Hom}_C(\mathfrak{J}^C), \\ &a, b \in \text{Hom}_C(\mathfrak{J}^C, C), \\ &M, N \in \mathfrak{J}^C, \\ &\mu, \nu \in C. \end{aligned}$$

The condition $\alpha(0, 0, 1, 0), \alpha(0, 0, 0, 1) \in \mathfrak{M}^C$ implies that

$$\mu M = 0, \quad \nu N = 0.$$

We shall first show that $M = N = 0$. Suppose that $M \neq 0, \mu = 0$. Then, the condition $\{\alpha(0, 0, 1, 0), \alpha(0, 0, 0, 1)\} = \{(0, 0, 1, 0), (0, 0, 0, 1)\} = 1$ implies that

$$(M, N) = 1. \tag{i}$$

Hence we have $N \neq 0, \nu = 0$. From

$$\mathfrak{M}^C \ni \alpha \begin{pmatrix} X \\ \frac{1}{\eta} X \times X \\ \frac{1}{\eta^2} \det X \\ \eta \end{pmatrix} = \begin{pmatrix} \beta X + \frac{1}{\eta^2} (\det X) M \\ \frac{1}{\eta} \delta(X \times X) + \eta N \\ a(X) \\ \frac{1}{\eta} b(X \times X) \end{pmatrix},$$

we have

$$\begin{aligned} \left(\frac{1}{\eta} \delta(X \times X) + \eta N \right) \times \left(\frac{1}{\eta} \delta(X \times X) + \eta N \right) &= a(X) \left(\beta X + \frac{1}{\eta^2} (\det X) M \right), \\ \left(\beta X + \frac{1}{\eta^2} (\det X) M, \frac{1}{\eta} \delta(X \times X) + \eta N \right) &= 3a(X) \frac{1}{\eta} b(X \times X) \end{aligned}$$

hold for all $0 \neq \eta \in C$. Comparing the coefficients of η , we have

$$\begin{cases} 2\delta(X \times X) \times N = a(X)\beta X & \text{(ii)} \\ \delta(X \times X) \times \delta(X \times X) = a(X)(\det X)M & \text{(iii)} \\ (\beta X, \delta(X \times X)) + \det X = 3a(X)b(X \times X) \text{ (use (i)).} & \text{(iv)} \end{cases}$$

Therefore, using (i) \sim (iv), we have

$$\begin{aligned} a(X)\det X &= a(X)(\det X)(M, N) = (\delta(X \times X) \times \delta(X \times X), N) \\ &= (\delta(X \times X), \delta(X \times X) \times N) = \frac{1}{2}a(X)(\delta(X \times X), \beta X) \\ &= \frac{1}{2}a(X)(3a(X)b(X \times X) - \det X). \end{aligned}$$

Hence we have $a(X)\det X = a(X)^2b(X \times X)$. Furthermore we have

$$\det X = a(X)b(X \times X). \quad (\text{v})$$

Indeed, from $\mu = 0$, we deduce that $a \neq 0$. Since $a : \mathfrak{J}^C \rightarrow C$ is a linear form, the set $\{X \in \mathfrak{J}^C \mid a(X) \neq 0\}$ is dense in \mathfrak{J}^C and the correspondence $\det X$ and $b(X \times X)$ is continuous with respect to X , (v) is also valid for $X \in \mathfrak{J}^C$ such that $a(X) = 0$. Now, since $a \neq 0$ and $b \neq 0$, (v) contradicts the irreducibility of the determinant $\det X$ with respect to the variables of its components. Consequently we have shown that $M = 0$. Similarly we can prove that $N = 0$. Therefore $\alpha(0, 0, 1, 0) = (0, 0, \mu, 0)$, $\alpha(0, 0, 0, 1) = (0, 0, 0, \nu)$. From the condition $\{\alpha \dot{1}, \alpha \dot{1}\} = 1$, $\langle \alpha \dot{1}, \alpha \dot{1} \rangle = 1$, we deduce that

$$\alpha(0, 0, 1, 0) = (0, 0, \mu, 0), \quad \alpha(0, 0, 0, 1) = (0, 0, 0, \mu^{-1}), \quad \mu \in C, (\tau\mu)\mu = 1.$$

If we choose $\theta \in C$ such that $\theta^3 = \mu$ and let $\beta = \varphi_1(\theta)^{-1}\alpha$, then $\beta(0, 0, 1, 0) = (0, 0, 1, 0)$, $\beta(0, 0, 0, 1) = (0, 0, 0, 1)$. Hence, $\beta \in E_6$ (Theorem 4.7.2) and we have

$$\alpha = \varphi_1(\theta)\beta, \quad \theta \in U(1), \beta \in E_6.$$

This shows φ is onto. That $\text{Ker } \varphi = \{(1, 1), (\omega, \omega 1), (\omega^2, \omega^2 1)\} = \mathbf{Z}_3$ is easily obtained. Thus we have the isomorphism $(U(1) \times E_6)/\mathbf{Z}_3 \cong (E_7)^\iota$.

Remark. $(E_7)^\iota$ is connected as a fixed points subgroup under the involution ι of the simply connected Lie group E_7 . Hence, to show that $\varphi : U(1) \times E_6 \rightarrow (E_6)^\iota$ is onto, it is sufficient to show that $\varphi_* : \mathfrak{u}(1) \oplus \mathfrak{e}_6 \rightarrow (\mathfrak{e}_7)^\iota$ is onto, which is easily shown.

4.11. Involution σ and subgroup $(SU(2) \times Spin(12))/\mathbf{Z}_2$ of E_7

We define an involutive C -linear transformation σ of \mathfrak{P}^C by

$$\sigma(X, Y, \xi, \eta) = (\sigma X, \sigma Y, \xi, \eta).$$

which is the extension of the C -linear transformation σ of \mathfrak{J}^C . This is the same as $\sigma \in F_4$ regarding as $\sigma \in F_4 \subset E_6 \subset E_7$.

We shall now study the following subgroup $(E_7)^\sigma$ of E_7 :

$$(E_7)^\sigma = \{\alpha \in E_7 \mid \sigma\alpha = \alpha\sigma\}.$$

To this end, we define two C -linear mappings $\kappa, \mu : \mathfrak{P}^C \rightarrow \mathfrak{P}^C$ by

$$\kappa = \Phi(-2E_1 \vee E_1, 0, 0, -1), \quad \mu = \Phi(0, E_1, E_1, 0).$$

The explicit forms of κ and μ are given by

$$\begin{aligned} \kappa \begin{pmatrix} X \\ Y \\ \xi \\ \eta \end{pmatrix} &= \begin{pmatrix} -\kappa_1 X \\ \kappa_1 Y \\ -\xi \\ \eta \end{pmatrix}, \quad \kappa_1 X = (E_1, X) - 4E_1 \times (E_1 \times X), \\ \mu \begin{pmatrix} X \\ Y \\ \xi \\ \eta \end{pmatrix} &= \begin{pmatrix} 2E_1 \times Y + \eta E_1 \\ 2E_1 \times X + \xi E_1 \\ (E_1, Y) \\ (E_1, X) \end{pmatrix}. \end{aligned}$$

More precisely, κ and μ are of the form

$$\begin{aligned} \kappa(X, Y, \xi, \eta) &= \kappa \left(\begin{pmatrix} \xi_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & \xi_2 & x_1 \\ x_2 & \bar{x}_1 & \xi_3 \end{pmatrix}, \begin{pmatrix} \eta_1 & y_3 & \bar{y}_2 \\ \bar{y}_3 & \eta_2 & y_1 \\ y_2 & \bar{y}_1 & \eta_3 \end{pmatrix}, \xi, \eta \right) \\ &= \left(\begin{pmatrix} -\xi_1 & 0 & 0 \\ 0 & \xi_2 & x_1 \\ 0 & \bar{x}_1 & \xi_3 \end{pmatrix}, \begin{pmatrix} \eta_1 & 0 & 0 \\ 0 & -\eta_2 & -y_1 \\ 0 & -\bar{y}_1 & -\eta_3 \end{pmatrix}, -\xi, \eta \right), \\ \mu(X, Y, \xi, \eta) &= \left(\begin{pmatrix} \eta & 0 & 0 \\ 0 & \eta_3 & -y_1 \\ 0 & -\bar{y}_1 & \eta_2 \end{pmatrix}, \begin{pmatrix} \xi & 0 & 0 \\ 0 & \xi_3 & -x_1 \\ 0 & -\bar{x}_1 & \xi_2 \end{pmatrix}, \eta_1, \xi_1 \right). \end{aligned}$$

$$\textbf{Lemma 4.11.1.} \quad (1) \quad \kappa\mu = \mu\kappa, \quad \begin{cases} \kappa\sigma = \sigma\kappa \\ \mu\sigma = \sigma\mu, \end{cases} \quad \begin{cases} \kappa\lambda = -\lambda\kappa \\ \mu\lambda = -\lambda\mu. \end{cases}$$

(2) If $\alpha \in E_7$ satisfies $\kappa\alpha = \alpha\kappa$, then α also satisfies $\sigma\alpha = \alpha\sigma$.

Proof. (1) These are checked by direct calculations.

(2) Since $\sigma = \exp \pi i \kappa$, we have $\sigma\alpha = (\exp \pi i \kappa)\alpha = \alpha(\exp \pi i \kappa) = \alpha\sigma$.

We shall first study the following subgroups $(E_7)^{\kappa, \mu}$ and $((E_7)^{\kappa, \mu})_{(0, E_1, 0, 1)}$ of E_7 :

$$\begin{aligned} (E_7)^{\kappa, \mu} &= \{\alpha \in E_7 \mid \kappa\alpha = \alpha\kappa, \mu\alpha = \alpha\mu\}, \\ ((E_7)^{\kappa, \mu})_{(0, E_1, 0, 1)} &= \{\alpha \in (E_7)^{\kappa, \mu} \mid \alpha(0, E_1, 0, 1) = (0, E_1, 0, 1)\}. \end{aligned}$$

Proposition 4.11.2. *The Lie algebras $(\mathfrak{e}_7)^\sigma$, $(\mathfrak{e}_7)^{\kappa, \mu}$, $((\mathfrak{e}_7)^{\kappa, \mu})_{(0, E_1, 0, 1)}$ of the groups $(E_7)^\sigma$, $(E_7)^{\kappa, \mu}$, $((E_7)^{\kappa, \mu})_{(0, E_1, 0, 1)}$ are respectively given by*

$$\begin{aligned} (1) \quad (\mathfrak{e}_7)^\sigma &= \{\Phi \in \mathfrak{e}_7 \mid \sigma\Phi = \Phi\sigma\} \\ &= \{\Phi(\phi, A, -\tau A, \nu) \in \mathfrak{e}_7 \mid \phi \in (\mathfrak{e}_6)^\sigma, A \in (\mathfrak{J}^C)_\sigma\}. \\ (2) \quad (\mathfrak{e}_7)^{\kappa, \mu} &= \{\Phi \in \mathfrak{e}_7 \mid \kappa\Phi = \Phi\kappa, \mu\Phi = \Phi\mu\} \\ &= \left\{ \Phi(\phi, A, -\tau A, \nu) \in \mathfrak{e}_7 \left| \begin{array}{l} \phi \in (\mathfrak{e}_6)^\sigma, A \in (\mathfrak{J}^C)_\sigma, (E_1, A) = 0, \\ \nu = -\frac{3}{2}(\phi E_1, E_1) \end{array} \right. \right\}. \\ (3) \quad ((\mathfrak{e}_7)^{\kappa, \mu})_{(0, E_1, 0, 1)} &= \{\Phi \in (\mathfrak{e}_7)^{\kappa, \mu} \mid \Phi((0, E_1, 0, 1)) = 0\} \\ &= \left\{ \Phi(\phi, A, -\tau A, 0) \in \mathfrak{e}_7 \left| \begin{array}{l} \phi \in \mathfrak{e}_6, \phi E_1 = 0, \\ A \in \mathfrak{J}^C, 2E_1 \times A = \tau A \end{array} \right. \right\}. \end{aligned}$$

Proof. (1) It is not difficult to prove and so is omitted here.

(2) Suppose that $\Phi = \Phi(\phi, A, -\tau A, \nu) \in \mathfrak{e}_7$ satisfies $\kappa\Phi = \Phi\kappa$ and $\mu\Phi = \Phi\mu$. Comparing the ξ -term of $\kappa\Phi P = \Phi\kappa P$, $P = (X, Y, \xi, \eta) \in \mathfrak{P}^C$, we have

$$(A, Y) = -(A, \kappa_1 Y), \quad Y \in \mathfrak{J}^C.$$

Let $Y = E_1$, then we have $(A, E_1) = 0$. Next, comparing the η -term of $\mu\Phi = \Phi\mu$, we have

$$(E_1, \phi X) - \frac{1}{3}\nu(E_1, X) = -\nu(E_1, X). \quad (i)$$

Since $\phi \in (\mathfrak{e}_6)^\sigma$, we can set $\phi E_1 = kE_1$, $k \in i\mathbf{R}$ (Lemma 3.10.1). Hence let $X = E_1$ in (i), then we have $k = -\frac{2}{3}\nu$. Conversely, if $\Phi = \Phi(\phi, A, -\tau A, \nu) \in \mathfrak{e}_7$ has the condition above, then from the following Lemma 4.11.3, we can see that Φ satisfies $\kappa\Phi = \Phi\kappa$ and $\mu\Phi = \Phi\mu$.

Lemma 4.11.3. *In \mathfrak{J}^C , the following hold.*

- (1) For $A \in (\mathfrak{J}^C)_\sigma$, we have $\kappa_1(A \times X) = \kappa_1 A \times \kappa_1 X$, $X \in \mathfrak{J}^C$.
- (2) For $\phi \in (\mathfrak{e}_6)^\sigma$, we have $\kappa_1 \phi = \phi \kappa_1$.
- (3) For $A \in (\mathfrak{J}^C)_\sigma$, $(E_1, A) = 0$, we have $\kappa_1 A = -A$ and
$$-4\tau A \times (E_1 \times X) + (E_1, X)A = 4E_1 \times (A \times X) - \langle A, X \rangle E_1, \quad X \in \mathfrak{J}^C.$$

We shall now return to the proof of (3) of Proposition 4.11.2.

If $\Phi = \Phi(\phi, A, -\tau A, \nu) \in (\mathfrak{e}_7)^{\kappa, \mu}$ satisfies $\Phi((0, E_1, 0, 1)) = 0$, then

$$\nu = -(A, E_1) = -\tau(A, E_1),$$

so that $\nu = \tau\nu$. Together with $\tau\nu = -\nu$, we have $\nu = 0$ and $\phi E_1 = 0$, furthermore, we have $2A \times E_1 = \tau A$ (in this case, $A \in (\mathfrak{J}^C)_\sigma$ and $(E_1, A) = 0$ automatically hold).

For $\nu \in i\mathbf{R}$, we define a C -linear mapping $\phi(\nu) : \mathfrak{J}^C \rightarrow \mathfrak{J}^C$ by

$$\phi(\nu) = 2\nu E_1 \vee E_1,$$

that is,

$$\phi(\nu) \begin{pmatrix} \xi_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & \xi_2 & x_1 \\ x_2 & \bar{x}_1 & \xi_3 \end{pmatrix} = \frac{\nu}{3} \begin{pmatrix} 4\xi_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & -2\xi_2 & -2x_1 \\ x_2 & -2\bar{x}_1 & -2\xi_3 \end{pmatrix}$$

(Lemma 3.4.2.(2)). Then $\phi(\nu) \in (\mathfrak{e}_6)^\sigma$.

Proposition 4.11.4. (1) $\mathfrak{a}_1 = \{\Phi(\phi(\nu), aE_1, -\tau aE_1, \nu) \mid a \in C, \nu \in i\mathbf{R}\}$ is a Lie subalgebra of $(\mathfrak{e}_7)^\sigma$ and isomorphic to the Lie algebra $\mathfrak{su}(2)$.

(2) The Lie algebra $(\mathfrak{e}_7)^\sigma$ is isomorphic to the direct sum of Lie algebras \mathfrak{a}_1 and $(\mathfrak{e}_7)^{\kappa, \mu}$:

$$(\mathfrak{e}_7)^\sigma \cong \mathfrak{a}_1 \oplus (\mathfrak{e}_7)^{\kappa, \mu}.$$

Proof. (1) The mapping $\varphi_* : \mathfrak{a}_1 \rightarrow \mathfrak{su}(2) = \{D \in M(2, C) \mid \tau({}^t D) = -D\}$ defined by

$$\varphi_*(\Phi(\phi(\nu), aE_1, -\tau aE_1, \nu)) = \begin{pmatrix} \nu & a \\ -\tau a & -\nu \end{pmatrix}$$

gives an isomorphism as Lie algebras. Indeed, this is clear from

$$\left[\begin{pmatrix} \nu & a \\ -\tau a & -\nu \end{pmatrix}, \begin{pmatrix} \rho & b \\ -\tau b & -\rho \end{pmatrix} \right] = \begin{pmatrix} b(\tau a) - a(\tau b) & 2(b\nu - a\rho) \\ -2\tau(b\nu - a\rho) & a(\tau b) - b(\tau a) \end{pmatrix},$$

and

$$\begin{aligned} & \left[\Phi(\phi(\nu), aE_1, -\tau aE_1, \nu), \Phi(\phi(\rho), bE_1, -\tau bE_1, \rho) \right] \\ &= \Phi(\phi(b\tau a) - a(\tau b), 2(b\nu - a\rho)E_1, -2\tau(b\nu - a\rho)E_1, (\tau a)b - a(\tau b)). \end{aligned}$$

(2) Using (1) above and Proposition 4.11.2.(2), the following decomposition of $(\mathfrak{e}_7)^\sigma$,

$$(\mathfrak{e}_7)^\sigma \ni \Phi \begin{pmatrix} \phi \\ A \\ -\tau A \\ \nu \end{pmatrix} = \Phi \begin{pmatrix} \phi(\nu') \\ aE_1 \\ -\tau aE_1 \\ \nu' \end{pmatrix} + \Phi \begin{pmatrix} \phi - \phi(\nu') \\ A - aE_1 \\ -\tau A + \tau aE_1 \\ \nu - \nu' \end{pmatrix} \in \mathfrak{a}_1 \oplus (\mathfrak{e}_7)^{\kappa, \mu},$$

where $\nu' = \frac{1}{3}\nu + \frac{1}{2}(E_1, \phi E_1)$, $a = (E_1, A)$, gives an isomorphism as Lie algebras.

Lemma 4.11.5. *For $a \in C$, we have*

$$\alpha_{23}(a) = \alpha_2(a)\alpha_3(\tau a) \in ((E_7)^{\kappa,\mu})_{(0,E_1,0,1)},$$

where $\alpha_i(a) \in E_7, i = 2, 3$ are defined in Lemma 4.8.1.

Proof. Since $\Phi(0, -\tau a E_i, a E_i, 0) \in (\mathfrak{e}_7)^{\kappa,\mu}$, we have $\alpha_i(a) = \exp \Phi(0, -\tau a E_i, a E_i, 0) \in (E_7)^{\kappa,\mu}, i = 2, 3$. Since $\alpha_2(a)$ and $\alpha_3(\tau a)$ are commutative, we have

$$\begin{aligned} \alpha_{23}(a) &= \alpha_2(a)\alpha_3(\tau a) \\ &= \exp \Phi(0, -\tau a E_2 - a E_3, a E_2 + \tau a E_3, 0). \end{aligned}$$

Since $\Phi(0, -\tau a E_2 - a E_3, a E_2 + \tau a E_3, 0) \in ((\mathfrak{e}_7)^{\kappa,\mu})_{(0,E_1,0,1)}$, we have $\alpha_{23}(a) \in ((E_7)^{\kappa,\mu})_{(0,E_1,0,1)}$.

We recall the group

$$\begin{aligned} Spin(10) &= \{\alpha \in E_6 \mid \alpha E_1 = E_1\} \\ &= \{\alpha \in E_6 \mid \sigma \alpha = \alpha \sigma, \alpha E_1 = E_1\} \subset E_7 \end{aligned}$$

(Lemma 3.10.4). The group $Spin(10)$ acts transitively on the 9 dimensional sphere

$$S^9 = \left\{ \left(\begin{pmatrix} 0 & 0 & 0 \\ 0 & \xi & x \\ 0 & \bar{x} & -\tau \xi \end{pmatrix}, 0, 0, 0 \right) \mid \xi \in C, x \in \mathfrak{C}, \bar{x}x + (\tau \xi)\xi = 1 \right\}.$$

Lemma 4.11.6. *For $\alpha \in ((E_7)^{\kappa,\mu})_{(0,E_1,0,1)}$, we have*

$$\alpha(0, -E_1, 0, 1) = (0, -E_1, 0, 1) \quad \text{if and only if} \quad \alpha(0, 0, 1, 0) = (0, 0, 1, 0).$$

In particular, we have

$$\{\alpha \in ((E_7)^{\kappa,\mu})_{(0,E_1,0,1)} \mid \alpha(0, -E_1, 0, 1) = (0, -E_1, 0, 1)\} \cong Spin(10).$$

Proof. If $\alpha \in (E_7)^{\kappa,\mu}$ satisfies $\alpha(0, E_1, 0, 1) = (0, E_1, 0, 1)$ and $\alpha(0, -E_1, 0, 1) = (0, -E_1, 0, 1)$, then we have $\alpha(0, 0, 0, 1) = (0, 0, 0, 1)$ and $\alpha(0, E_1, 0, 0) = (0, E_1, 0, 0)$, which imply that $\alpha(0, 0, 1, 0) = \alpha\mu(0, E_1, 0, 0) = \mu\alpha(0, E_1, 0, 0) = \mu(0, E_1, 0, 0) = (0, 0, 1, 0)$. The converse can be similarly proved. If $\alpha \in E_7$ satisfies $\alpha(0, 0, 1, 0) = (0, 0, 1, 0)$, then $\alpha \in E_6$ (Theorem 4.7.2), and from the condition $\alpha E_1 = E_1$, we obtain $\alpha \in Spin(10)$, (Theorem 3.10.4). The converse also holds.

We define an 11 dimensional \mathbf{R} -vector space V^{11} by

$$\begin{aligned} V^{11} &= \{P \in \mathfrak{P}^C \mid \kappa P = P, \mu\tau\lambda P = P, P \times (0, E_1, 0, 1, 0) = 0\} \\ &= \left\{ \left(\begin{pmatrix} 0 & 0 & 0 \\ 0 & \xi & x \\ 0 & \bar{x} & -\tau \xi \end{pmatrix}, \begin{pmatrix} \eta & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, 0, \tau \eta \right) \mid x \in \mathfrak{C}, \xi \in C, \eta \in i\mathbf{R} \right\} \end{aligned}$$

with the norm $(P, P)_\mu$ given by

$$(P, P)_\mu = \frac{1}{2} \{ \mu P, P \} = \bar{x}x + (\tau\xi)\xi + (\tau\eta)\eta.$$

Proposition 4.11.7. $((E_7)^{\kappa, \mu})_{(0, E_1, 0, 1)} / Spin(10) \simeq S^{10}$.

In particular, the group $((E_7)^{\kappa, \mu})_{(0, E_1, 0, 1)}$ is connected.

Proof. $S^{10} = \{P \in V^{11} \mid (P, P)_\mu = 1\}$ is a 10 dimensional sphere. For $\alpha \in ((E_7)^{\kappa, \mu})_{(0, E_1, 0, 1)}$ and $P \in S^{10}$, we have $\alpha P \in S^{10}$ (Proposition 4.2.2, Lemma 4.3.3). Hence the group $((E_7)^{\kappa, \mu})_{(0, E_1, 0, 1)}$ acts on S^{10} . We shall prove that this action is transitive. To prove this, it is sufficient to show that any element $P \in S^{10}$ can be transformed to $(0, -iE_1, 0, i) \in S^{10}$ by some $\alpha \in ((E_7)^{\kappa, \mu})_{(0, E_1, 0, 1)}$. Now, for a given

$$P = \left(\begin{pmatrix} 0 & 0 & 0 \\ 0 & \xi & x \\ 0 & \bar{x} & -\tau\xi \end{pmatrix}, \begin{pmatrix} \eta & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, 0, \tau\eta \right) \in S^{10},$$

choose $a \in \mathbf{R}$, $0 \leq a < \frac{\pi}{4}$, such that

$$\tan 2a = \frac{2\eta}{\tau\xi - \xi}.$$

(If $\tau\xi - \xi = 0$, then we choose $a = \frac{\pi}{4}$). Applying $\alpha_{23}(a)$ of Lemma 4.11.5 on P , then the η -part of $\alpha_{23}(a)P$ becomes

$$\begin{aligned} & 2 \sin^2 a (E_2, E_3 \times X) + \tau\xi \sin a \cos a - (E_3, Y) \sin a \cos a - \eta \cos^2 a \\ &= \eta \sin^2 a + (\tau\xi - \xi) \sin a \cos a - \eta \cos^2 a \\ &= \frac{1}{2}(\tau\xi - \xi) \sin 2a - \eta \cos 2a = 0. \end{aligned}$$

Hence we have

$$\alpha_{23}(a)P \in S^9.$$

Since the group $Spin(10)$ acts transitively on S^9 (Proposition 3.10.3), there exists $\beta \in Spin(10) = (E_6)_{E_1} \subset ((E_7)^{\kappa, \mu})_{(0, E_1, 0, 1)}$ such that

$$\beta \alpha_{23}(a)P = (i(E_2 + E_3), 0, 0, 0).$$

Again, applying $\alpha_{23}(-\pi/4) \in ((E_7)^{\kappa, \mu})_{(0, E_1, 0, 1)}$ of Lemma 4.11.5 on the above, we have

$$\alpha_{23}(-\pi/4) \beta \alpha_{23}(a)P = (0, -iE_1, 0, i).$$

This shows the transitivity of $((E_7)^{\kappa, \mu})_{(0, E_1, 0, 1)}$. The isotropy subgroup of $((E_7)^{\kappa, \mu})_{(0, E_1, 0, 1)}$ at $(0, -iE_1, 0, i)$ is $Spin(10)$ (Lemma 4.11.6). Thus we have the homeomorphism $((E_7)^{\kappa, \mu})_{(0, E_1, 0, 1)} / Spin(10) \simeq S^{10}$.

Theorem 4.11.8. $((E_7)^{\kappa, \mu})_{(0, E_1, 0, 1)} \cong Spin(11)$.

(From now on, we identify these groups).

Proof. Analogous to Theorem 3.10.4, we can define a homomorphism $p : ((E_7)^{\kappa, \mu})_{(0, E_1, 0, 1)} \rightarrow SO(11) = SO(V^{11})$ by $p(\alpha) = \alpha|V^{11}$. The restriction p' of p to $(E_6)_{E_1}$ coincides with the homomorphism $p' : Spin(10) \rightarrow SO(10) = SO(V^{10})$ (where $V^{10} = \{P \in V^{11} \mid P = (X, 0, 0, 0)\}$). In particular, $p' : Spin(10) \rightarrow SO(10)$ is onto. Hence, from the following commutative diagram

$$\begin{array}{ccccccc} 1 & \longrightarrow & Spin(10) & \longrightarrow & ((E_7)^{\kappa, \mu})_{(0, E_1, 0, 1)} & \longrightarrow & S^{10} \longrightarrow * \\ & & \downarrow p' & & \downarrow p & & \downarrow = \\ 1 & \longrightarrow & SO(10) & \longrightarrow & SO(11) & \longrightarrow & S^{10} \longrightarrow * \end{array}$$

we see that $p : ((E_7)^{\kappa, \mu})_{(0, E_1, 0, 1)} \rightarrow SO(11)$ is onto by the five lemma. Using the five lemma again, we see that $\text{Ker } p$ coincides with $\text{Ker } p'$. Hence $\text{Ker } p = \{1, \sigma\}$ (Theorem 3.10.4). Thus we have the isomorphism

$$((E_7)^{\kappa, \mu})_{(0, E_1, 0, 1)} / \{1, \sigma\} \cong SO(11).$$

Therefore the group $((E_7)^{\kappa, \mu})_{(0, E_1, 0, 1)}$ is isomorphic to the group $Spin(11)$ as the universal covering group of $SO(11)$.

Lemma 4.11.9. For $t \in \mathbf{R}$, we define a mapping $\alpha(t) : \mathfrak{P}^C \rightarrow \mathfrak{P}^C$ by

$$\begin{aligned} \alpha(t) & \left(\begin{pmatrix} \xi_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & \xi_2 & x_1 \\ x_2 & \bar{x}_1 & \xi_3 \end{pmatrix}, \begin{pmatrix} \eta_1 & y_3 & \bar{y}_2 \\ \bar{y}_3 & \eta_2 & y_1 \\ y_2 & \bar{y}_1 & \eta_3 \end{pmatrix}, \xi, \eta \right) \\ & = \left(\begin{pmatrix} e^{2it}\xi_1 & e^{it}x_3 & e^{it}\bar{x}_2 \\ e^{it}\bar{x}_3 & \xi_2 & x_1 \\ e^{it}x_2 & \bar{x}_1 & \xi_3 \end{pmatrix}, \begin{pmatrix} e^{-2it}\eta_1 & e^{-it}y_3 & e^{-it}\bar{y}_2 \\ e^{-it}\bar{y}_3 & \eta_2 & y_1 \\ e^{-it}y_2 & \bar{y}_1 & \eta_3 \end{pmatrix}, e^{-2it}\xi, e^{2it}\eta \right), \end{aligned}$$

then $\alpha(t) \in (E_7)^{\kappa, \mu}$.

Proof. For $\nu = it \in i\mathbf{R}$, let $\phi(\nu) = 2\nu E_1 \vee E_1 \in (\mathfrak{e}_6)^\sigma$. Then, $\Phi(\phi(\nu), 0, 0, -2\nu) \in (\mathfrak{e}_7)^{\kappa, \nu}$ (Proposition 4.11.2) and $\alpha(t) = \exp \Phi(\phi(\nu), 0, 0, -2\nu)$. Hence we have $\alpha(t) \in (E_7)^{\kappa, \mu}$.

We define a 12 dimensional \mathbf{R} -vector space V^{12} by

$$\begin{aligned} V^{12} & = \{P \in \mathfrak{P}^C \mid \kappa P = P, \mu \tau \lambda P = P\} \\ & = \left\{ \left(\begin{pmatrix} 0 & 0 & 0 \\ 0 & \xi & x \\ 0 & \bar{x} & -\tau \xi \end{pmatrix}, \begin{pmatrix} \eta & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, 0, \tau \eta \right) \mid x \in \mathfrak{C}, \xi, \eta \in C \right\} \end{aligned}$$

with the norm $(P, P)_\mu$ given by

$$(P, P)_\mu = \frac{1}{2} \{\mu P, P\} = \bar{x}x + (\tau \xi)\xi + (\tau \eta)\eta.$$

Proposition 4.11.10. $(E_7)^{\kappa, \mu} / Spin(11) \simeq S^{11}$.

In particular, the group $(E_7)^{\kappa,\mu}$ is connected.

Proof. $S^{11} = \{P \in V^{12} \mid (P, P)_\mu = 1\}$ is an 11 dimensional sphere. For $\alpha \in (E_7)^{\kappa,\mu}$ and $P \in S^{11}$, we have $\alpha P \in S^{11}$ (Proposition 4.2.2, Lemma 4.3.3). Hence the group $(E_7)^{\kappa,\mu}$ acts on S^{11} . We shall prove that this action is transitive. To prove this, it is sufficient to show that any element $P \in S^{11}$ can be transformed to $(0, E_1, 0, 1) \in S^{11}$ by some $\alpha \in (E_7)^{\kappa,\mu}$. Now, for a given

$$P = \left(\begin{pmatrix} 0 & 0 & 0 \\ 0 & \xi & x \\ 0 & \bar{x} & -\tau\xi \end{pmatrix}, \begin{pmatrix} \eta & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, 0, \tau\eta \right) \in S^{11},$$

we choose $t \in \mathbf{R}$ such that $e^{-2it}\eta \in i\mathbf{R}$. Applying $\alpha(t)$ of Lemma 4.11.9 on P , we get

$$\alpha(t)P \in S^{10}.$$

Since the group $Spin(11)$ acts transitively on S^{10} (Proposition 4.11.7), there exists $\beta \in Spin(11) = ((E_7)^{\kappa,\mu})_{(0, E_1, 0, 1)}$ such that

$$\beta\alpha(t)P = (0, -iE_1, 0, i).$$

If we further apply $\alpha(-\pi/4) \in (E_7)^{\kappa,\mu}$ of Lemma 4.11.9 on the above, then we have

$$\alpha(-\pi/4)\beta\alpha(t)P = (0, E_1, 0, 1).$$

This shows the transitivity of $(E_7)^{\kappa,\mu}$. The isotropy subgroup of $(E_7)^{\kappa,\mu}$ at $(0, E_1, 0, 1)$ is $Spin(11)$ (Theorem 4.11.8). Thus we have the homeomorphism $(E_7)^{\kappa,\mu} / Spin(11) \simeq S^{11}$.

Theorem 4.11.11. $(E_7)^{\kappa,\mu} \cong Spin(12)$.

(From now on, we identify these groups).

Proof. Analogous to Theorem 4.11.8, we can define a homomorphism

$$p : (E_7)^{\kappa,\mu} \rightarrow SO(12) = SO(V^{12})$$

by $p(\alpha) = \alpha|V^{12}$. The restriction p' of p to $(E_6)^{\kappa,\mu}$ coincides with the homomorphism $p' : Spin(11) \rightarrow SO(11)$ of Theorem 4.11.8. In particular, $p' : Spin(11) \rightarrow SO(11)$ is onto. Hence from the following commutative diagram

$$\begin{array}{ccccccc} 1 & \longrightarrow & Spin(11) & \longrightarrow & (E_7)^{\kappa,\mu} & \longrightarrow & S^{11} \longrightarrow * \\ & & \downarrow p' & & \downarrow p & & \downarrow = \\ 1 & \longrightarrow & SO(11) & \longrightarrow & SO(12) & \longrightarrow & S^{11} \longrightarrow * \end{array}$$

we see that $p : (E_7)^{\kappa,\mu} \rightarrow SO(12)$ is onto by the five lemma. Using the five lemma again we see that $\text{Ker } p$ coincides with $\text{Ker } p'$. Hence $\text{Ker } p = \{1, \sigma\}$ (Theorem 4.11.8). Thus we have the isomorphism

$$(E_7)^{\kappa,\mu} / \{1, \sigma\} \cong SO(12).$$

Therefore the group $(E_7)^{\kappa, \mu}$ is isomorphic to the group $Spin(12)$ as the universal covering group of $SO(12)$.

Theorem 4.11.12. *The center $z(Spin(12))$ of $Spin(12)$ is*

$$z(Spin(12)) = \{1, -1, \sigma - \sigma\} \cong \{1, -1\} \times \{1, \sigma\} \cong \mathbf{Z}_2 \times \mathbf{Z}_2.$$

And we have

$$\begin{aligned} Spin(12)/\{1, \sigma\} &\cong SO(12), \\ Spin(12)/\{1, -1\} &\cong Spin(12)/\{1, -\sigma\} \cong Ss(12). \end{aligned}$$

Theorem 4.11.13. *The group $(E_7)^\sigma$ contains a subgroup*

$$\varphi_2(SU(2)) = \{\varphi_2(A) \in E_7 \mid A \in SU(2)\}$$

which is isomorphic to the special unitary group $SU(2) = \{A \in M(2, C) \mid {}^t(\tau A)A = E, \det A = 1\}$. Here, for $A \in SU(2)$, a mapping $\varphi_2(A) : \mathfrak{P}^C \rightarrow \mathfrak{P}^C$ is defined by

$$\begin{aligned} \varphi_2(A) &\left(\begin{pmatrix} \xi_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & \xi_2 & x_1 \\ x_2 & \bar{x}_1 & \xi_3 \end{pmatrix}, \begin{pmatrix} \eta_1 & y_3 & \bar{y}_2 \\ \bar{y}_3 & \eta_2 & y_1 \\ y_2 & \bar{y}_1 & \eta_3 \end{pmatrix}, \xi, \eta \right) \\ &= \left(\begin{pmatrix} \xi_1' & x_3' & \bar{x}_2' \\ \bar{x}_3' & \xi_2' & x_1' \\ x_2' & \bar{x}_1' & \xi_3' \end{pmatrix}, \begin{pmatrix} \eta_1' & y_3' & \bar{y}_2' \\ \bar{y}_3' & \eta_2' & y_1' \\ y_2' & \bar{y}_1' & \eta_3' \end{pmatrix}, \xi', \eta' \right), \end{aligned}$$

where

$$\begin{aligned} \begin{pmatrix} \xi_1' \\ \eta' \end{pmatrix} &= A \begin{pmatrix} \xi_1 \\ \eta \end{pmatrix}, \quad \begin{pmatrix} \xi' \\ \eta_1' \end{pmatrix} = A \begin{pmatrix} \xi \\ \eta_1 \end{pmatrix}, \quad \begin{pmatrix} \eta_2' \\ \xi_3' \end{pmatrix} = A \begin{pmatrix} \eta_2 \\ \xi_3 \end{pmatrix}, \quad \begin{pmatrix} \eta_3' \\ \xi_2' \end{pmatrix} = A \begin{pmatrix} \eta_3 \\ \xi_2 \end{pmatrix}, \\ \begin{pmatrix} x_1' \\ y_1' \end{pmatrix} &= \tau A \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \quad \begin{pmatrix} x_2' \\ y_2' \end{pmatrix} = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}, \quad \begin{pmatrix} x_3' \\ y_3' \end{pmatrix} = \begin{pmatrix} x_3 \\ y_3 \end{pmatrix}. \end{aligned}$$

Proof. The action of $\Phi(\phi(\nu), aE_1, -\tau aE_1, \nu) \in \mathfrak{a}_1$ ($\phi(\nu) = 2\nu E_1 \vee E_1, \nu \in i\mathbf{R}, a \in C$) on \mathfrak{P}^C is

$$\Phi(\phi(\nu), aE_1, -\tau aE_1, \nu)(X, Y, \xi, \eta) = (X', Y', \xi', \eta')$$

where

$$\begin{aligned} \begin{pmatrix} \xi_1' \\ \eta' \end{pmatrix} &= \begin{pmatrix} \nu & a \\ -\tau a & -\nu \end{pmatrix} \begin{pmatrix} \xi_1 \\ \eta \end{pmatrix}, \quad \begin{pmatrix} \xi' \\ \eta_1' \end{pmatrix} = \begin{pmatrix} \nu & a \\ -\tau a & -\nu \end{pmatrix} \begin{pmatrix} \xi \\ \eta_1 \end{pmatrix}, \\ \begin{pmatrix} \eta_2' \\ \xi_3' \end{pmatrix} &= \begin{pmatrix} \nu & a \\ -\tau a & -\nu \end{pmatrix} \begin{pmatrix} \eta_2 \\ \xi_3 \end{pmatrix}, \quad \begin{pmatrix} \eta_3' \\ \xi_2' \end{pmatrix} = \begin{pmatrix} \nu & a \\ -\tau a & -\nu \end{pmatrix} \begin{pmatrix} \eta_3 \\ \xi_2 \end{pmatrix}, \\ \begin{pmatrix} x_1' \\ y_1' \end{pmatrix} &= \begin{pmatrix} -\nu & \tau a \\ -a & \nu \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \quad \begin{pmatrix} x_2' \\ y_2' \end{pmatrix} = \begin{pmatrix} x_3' \\ y_3' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \end{aligned}$$

Therefore, for $A = \exp \begin{pmatrix} \nu & a \\ -\tau a & -\nu \end{pmatrix} \in SU(2)$, we have

$$\exp(\Phi(\phi(\nu), aE_1, -\tau aE_1, \nu)) = \varphi_2(A) \in \varphi_2(SU(2)) \subset (E_7)^\sigma.$$

Lemma 4.11.14. *The group $(E_7)^\sigma$ is connected.*

Proof. The group $(E_7)^\sigma$ is the fixed points set obtained by the involutive automorphism σ of the simply connected Lie group E_7 , hence $(E_7)^\sigma$ is connected.

Theorem 4.11.15. $(E_7)^\sigma \cong (SU(2) \times Spin(12))/Z_2$, $Z_2 = \{(E, 1), (-E, -\sigma)\}$.

Proof. We define a mapping $\varphi : SU(2) \times Spin(12) \rightarrow (E_7)^\sigma$ by

$$\varphi(A, \beta) = \varphi_2(A)\beta.$$

Since the Lie algebras \mathfrak{a}_1 and $(\mathfrak{e}_7)^{\kappa, \mu}$ of $SU(2)$ and $Spin(12)$ are elementwise commutative (Proposition 4.11.4.(2)), $\varphi_2(A) \in SU(2)$ and $\beta \in Spin(12)$ are commutative : $\varphi_2(A)\beta = \beta\varphi_2(A)$. Hence φ is a homomorphism. We shall show that φ is onto. Since the group $(E_7)^\sigma$ is connected (Lemma 4.11.14), to prove this, it is sufficient to show that its differential mapping $\varphi_* : \mathfrak{a}_1 \oplus (\mathfrak{e}_7)^{\kappa, \mu} \rightarrow (\mathfrak{e}_7)^\sigma$ is onto. However this has been already shown in Proposition 4.11.4.(2). $\text{Ker } \varphi = \{(E, 1), (-E, \varphi_2(-E))\} = \{(E, 1), (-E, -\sigma)\} = Z_2$ (Theorem 4.11.13) is easily obtained. Thus we have the isomorphism $(SU(2) \times Spin(12))/Z_2 \cong (E_7)^\sigma$.

Remark. We can give an elementary proof of Lemma 4.11.14 once we have proved the following three claims.

Claim 1. *Any element $X \in (\mathfrak{J}^C)_\sigma$ can be transformed to a diagonal form by some $\alpha \in (E_6)^\sigma$:*

$$\alpha X = \begin{pmatrix} \xi_1 & 0 & 0 \\ 0 & \xi_2 & 0 \\ 0 & 0 & \xi_3 \end{pmatrix}, \quad \xi_i \in C.$$

Moreover we can choose $\alpha \in (E_6)^\sigma$ so that $\xi_2 \geq 0, \xi_3 \geq 0$.

Proof. Recall that $i(E_1 - E_2)^\sim, i(E_1 - E_3)^\sim, i\tilde{F}_1(a), \tilde{A}_1(a) (a \in \mathfrak{C}) \in (\mathfrak{e}_6)^\sigma$, then we can prove analogously as in Proposition 3.8.2.

Claim 2. *Any element $P \in (\mathfrak{M}^C)_\sigma = \{P \in \mathfrak{M}^C \mid \sigma P = P\}$ can be transformed to a diagonal form by some $\alpha \in ((E_7)^\sigma)_0$ (the connected component of $(E_7)^\sigma$ containing the identity 1):*

$$\alpha P = (X, Y, \xi, \eta), \quad X, Y \text{ are diagonal, } \xi > 0.$$

Proof. Recall $\Phi(0, -\tau a E_i, a E_i, 0) \in (\mathfrak{e}_7)^\sigma, i = 1, 2, 3$, then we can prove analogously as in Proposition 4.8.2.

Claim 3. $(E_7)^\sigma / (E_6)^\sigma \simeq (\mathfrak{M}_1)_\sigma = \{P \in \mathfrak{M}_1 \mid \sigma P = P\}$.
In particular, the group $(E_7)^\sigma$ is connected.

Proof. Remark that $\alpha_i(a)$ of Lemma 4.8.1 belongs to $((E_7)^\sigma)_0$, then this claim is proved analogously as Theorem 4.8.3. The connectedness of $(E_7)^\sigma$ follows from the connectedness of $(E_6)^\sigma \cong (U(1) \times Spin(10))/Z_4$ (Theorem 3.10.7) and $(\mathfrak{M}_1)_\sigma$.

4.12. Involution $\tau\gamma$ and subgroup $SU(8)/\mathbf{Z}_2$ of E_7

We consider the involutive complex conjugate transformation $\tau\gamma$ of \mathfrak{P}^C ,

$$\tau\gamma(X, Y, \xi, \eta) = (\tau\gamma X, \tau\gamma Y, \tau\xi, \tau\eta),$$

where γ is the same as $\gamma \in G_2 \subset F_4 \subset E_6 \subset E_7$.

We shall study the following subgroup $(E_7)^{\tau\gamma}$ of E_7 :

$$\begin{aligned} (E_7)^{\tau\gamma} &= \{\alpha \in E_7 \mid \tau\gamma\alpha = \alpha\tau\gamma\} \\ &= \{\alpha \in E_7 \mid \lambda\gamma\alpha = \alpha\lambda\gamma\} = (E_7)^{\lambda\gamma}. \end{aligned}$$

To this end, we consider \mathbf{R} -vector subspaces $(\mathfrak{P}^C)_{\tau\gamma}$, $(\mathfrak{P}^C)_{-\tau\gamma}$ of \mathfrak{P}^C , which are eigenspaces of $\tau\gamma$, respectively by

$$\begin{aligned} (\mathfrak{P}^C)_{\tau\gamma} &= \{P \in \mathfrak{P}^C \mid \tau\gamma P = P\} \\ &= \{(X, Y, \xi, \eta) \in \mathfrak{P}^C \mid X, Y \in (\mathfrak{J}^C)_{\tau\gamma}, \xi, \eta \in \mathbf{R}\}, \\ (\mathfrak{P}^C)_{-\tau\gamma} &= \{P \in \mathfrak{P}^C \mid \tau\gamma P = -P\} \\ &= \{(X, Y, \xi, \eta) \in \mathfrak{P}^C \mid X, Y \in (\mathfrak{J}^C)_{-\tau\gamma}, \xi, \eta \in i\mathbf{R}\}, \\ &= i(\mathfrak{P}^C)_{\tau\gamma}. \end{aligned}$$

These spaces $(\mathfrak{P}^C)_{\tau\gamma}$, $(\mathfrak{P}^C)_{-\tau\gamma}$ are invariant under the action of $(E_7)^{\tau\gamma}$ and we have the decomposition

$$\mathfrak{P}^C = (\mathfrak{P}^C)_{\tau\gamma} \oplus (\mathfrak{P}^C)_{-\tau\gamma} = (\mathfrak{P}^C)_{\tau\gamma} \oplus i(\mathfrak{P}^C)_{\tau\gamma}.$$

In particular, \mathfrak{P}^C is the complexification of $(\mathfrak{P}^C)_{\tau\gamma}$: $\mathfrak{P}^C = ((\mathfrak{P}^C)_{\tau\gamma})^C$.

Analogous to Section 3.11, we can define the \mathbf{R} -linear mapping $k : M(4, \mathbf{H}) \rightarrow M(8, \mathbf{C})$,

$$k\left(\begin{pmatrix} a + be_2 \end{pmatrix}\right) = \begin{pmatrix} \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \end{pmatrix}, \quad a, b \in \mathbf{C}.$$

Lemma 4.12.1. *Any element $B \in \mathfrak{su}(8)$ is uniquely expressed by*

$$B = k(D) + e_1 k(T), \quad D \in \mathfrak{sp}(4), T \in \mathfrak{J}(4, \mathbf{H})_0.$$

Proof. For $B \in \mathfrak{su}(8)$, let $D_1 = \frac{B - J\bar{B}J}{2}$, $T_1 = \frac{B + J\bar{B}J}{2e_1} \in M(8, \mathbf{C})$, then we have

$$\begin{aligned} B &= D_1 + e_1 T_1, & D_1^* &= -D_1, JD_1 = \bar{D}_1 J, \\ & & T_1^* &= T_1, JT_1 = \bar{T}_1 J, \text{tr}(T_1) = 0. \end{aligned}$$

Then, $D = k^{-1}(D_1)$, $T = k^{-1}(T_1) \in M(4, \mathbf{H})$ are the required elements. To prove the uniqueness of the expression, it is sufficient to show that

$$D_1 + e_1 T_1 = 0, D_1 \in k(\mathfrak{sp}(4)), T_1 \in k(\mathfrak{J}(4, \mathbf{H})_0) \quad \text{implies} \quad D_1 = T_1 = 0.$$

Certainly, from the condition, we have $JD_1 + e_1JT_1 = 0$, so $\overline{D}_1J + e_1\overline{T}_1J = 0$, and so that $D_1J - e_1T_1J = 0$, that is, $D_1 - e_1T_1 = 0$. Together with the first equation, we have $D_1 = T_1 = 0$.

After this, we will use the C -linear mapping $g : \mathfrak{J}^C \rightarrow \mathfrak{J}(4, \mathbf{H})^C$, $g(M + \mathbf{a}) = \begin{pmatrix} \frac{1}{2}\text{tr}(M) & i\mathbf{a} \\ i\mathbf{a}^* & M - \frac{1}{2}\text{tr}(M)E \end{pmatrix}$, the homomorphism $\varphi : Sp(4) \rightarrow (E_6)^{\tau\gamma}$, $\varphi(A)X = g^{-1}(A(gX)A^*)$, $X \in \mathfrak{J}^C$ and its differential mapping $\varphi_* : \mathfrak{sp}(4) \rightarrow (\mathfrak{e}_6)^{\tau\gamma}$, $\varphi_*(D)X = g^{-1}(D(gX) + (gX)D^*)$, $X \in \mathfrak{J}^C$ which are defined in Section 3.12.

Proposition 4.12.2. *The Lie algebra $(\mathfrak{e}_7)^{\tau\gamma}$ of the group $(E_7)^{\tau\gamma}$ is*

$$\begin{aligned} (\mathfrak{e}_7)^{\tau\gamma} &= \{\Phi \in \mathfrak{e}_7 \mid \tau\gamma\Phi = \Phi\tau\gamma\} \\ &= \{\Phi(\phi, A, -\gamma A, 0) \in \mathfrak{e}_7 \mid \phi \in (\mathfrak{e}_6)^{\tau\gamma}, A \in (\mathfrak{J}^C)_{\tau\gamma}\} \\ &= \{\Phi(\varphi_*(D), g^{-1}(T), -\gamma g^{-1}(T), 0) \in \mathfrak{e}_7 \mid D \in \mathfrak{sp}(4), T \in \mathfrak{J}(4, \mathbf{H})_0\}. \end{aligned}$$

The Lie bracket $[\Phi_1, \Phi_2]$ in $(\mathfrak{e}_7)^{\tau\gamma}$ is given by

$$[\Phi(\phi_1, A_1, -\gamma A_1, 0), \Phi(\phi_2, A_2, -\gamma A_2, 0)] = \Phi(\phi, A, -\gamma A, 0),$$

where

$$\begin{cases} \phi = [\phi_1, \phi_2] - 2A_1 \vee \gamma A_2 + 2A_2 \vee \gamma A_1, \\ A = \phi_1 A_2 - \phi_2 A_1. \end{cases}$$

Proof. It is not difficult to verify them..

Analogous to Section 3.11, we define a C -vector space $\mathfrak{S}(8, C)$ by

$$\mathfrak{S}(8, C) = \{S \in M(8, C) \mid {}^tS = -S\},$$

and a C -linear mapping $k_J : \mathfrak{J}(4, \mathbf{H})^C \rightarrow \mathfrak{S}(8, C)^C$ by

$$k_J(M_1 + iM_2) = k(M_1)J + ik(M_2)J, \quad M_1, M_2 \in \mathfrak{J}(4, \mathbf{H}),$$

where $J = \text{diag}(J, J, J, J) \in M(8, C)$, $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Definition. We define a C -linear isomorphism $\chi : \mathfrak{P}^C \rightarrow \mathfrak{S}(8, C)^C$ by

$$\chi(X, Y, \xi, \eta) = k_J\left(gX - \frac{\xi}{2}E\right) + e_1k_J\left(g(\gamma Y) - \frac{\eta}{2}E\right).$$

Proposition 4.12.3. $(\mathfrak{e}_7)^{\tau\gamma} \cong \mathfrak{su}(8)$.

This isomorphism is given by the mapping $\varphi_* : \mathfrak{su}(8) \rightarrow (\mathfrak{e}_7)^{\tau\gamma}$,

$$\varphi_*(B)P = \chi^{-1}(B(\chi P) + (\chi P)^tB), \quad P \in \mathfrak{P}^C.$$

Proof. We first prove that for $B \in \mathfrak{su}(8)$ we have $\varphi_*(B) \in (\mathfrak{e}_7)^{\tau\gamma}$.

(i) For $B = k(D)$, $D \in \mathfrak{sp}(4)$, we have

$$\begin{aligned}
P &= (X, Y, \xi, \eta) \\
&\xrightarrow{\chi} k\left(gX - \frac{\xi}{2}E\right)J + e_1k\left(g(\gamma Y) - \frac{\eta}{2}E\right)J \\
&\longrightarrow k(D)k\left(gX - \frac{\xi}{2}E\right)J + e_1k(D)k\left(g(\gamma Y) - \frac{\eta}{2}E\right)J \\
&\quad + k\left(gX - \frac{\xi}{2}E\right)J^t k(D) + e_1k\left(g(\gamma Y) - \frac{\eta}{2}E\right)J^t k(D) \\
&= k\left(D\left(gX - \frac{\xi}{2}E\right)\right)J + e_1k\left(D\left(g(\gamma Y) - \frac{\eta}{2}E\right)\right)J \\
&\quad + k\left(\left(gX - \frac{\xi}{2}E\right)D^*\right)J + e_1k\left(\left(g(\gamma Y) - \frac{\eta}{2}E\right)D^*\right)J \\
&= k(D(gX) + (gX)D^*)J + e_1k(D(g(\gamma Y) + (g(\gamma Y))D^*))J \\
&= k(g(\varphi_*(D)X))J + e_1k(g(\varphi_*(D)(\gamma Y)))J \\
&\text{(recall } \varphi_*(C)X = g^{-1}(C(gX) + (gX)C^*)) \\
&= \chi\left(\begin{pmatrix} \varphi_*(D)X \\ \gamma\varphi_*(D)\gamma Y \\ 0 \\ 0 \end{pmatrix}\right) = \chi\left(\begin{pmatrix} \varphi_*(D)X & 0 & 0 & 0 \\ 0 & \tau\varphi_*(D)\tau & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} X \\ Y \\ \xi \\ \eta \end{pmatrix}\right) \\
&= \chi(\Phi(\varphi_*(D), 0, 0, 0)P).
\end{aligned}$$

Hence, we have $\varphi_*(k(D)) = \Phi(\varphi_*(D), 0, 0, 0) \in (\mathfrak{e}_7)^{\tau\gamma}$.

(ii) For $B = e_1k(T)$, $T \in \mathfrak{J}(4, \mathbf{H})_0$ (denote $T = gA$, $A \in (\mathfrak{J}^C)_{\tau\gamma}$)

$$\begin{aligned}
P &= (X, Y, \xi, \eta) \\
&\xrightarrow{\chi} k\left(gX - \frac{\xi}{2}E\right)J + e_1k\left(g(\gamma Y) - \frac{\eta}{2}E\right)J \\
&\longrightarrow e_1k(T)k\left(gX - \frac{\xi}{2}E\right)J - k(T)k\left(g(\gamma Y) - \frac{\eta}{2}E\right)J \\
&\quad + e_1k\left(gX - \frac{\xi}{2}E\right)J^t k(T) - k\left(g(\gamma Y) - \frac{\eta}{2}E\right)J^t k(T) \\
&= k(-Tg(\gamma Y) - g(\gamma Y)T + \eta T)J + e_1k(T(gX) + (gX)T - \xi T)J \\
&= k(-2gA \circ g(\gamma Y) + \eta gA)J + e_1k(2gA \circ gX - \xi gA)J \\
&= k\left(-2g(\gamma A \times Y) - \frac{1}{2}(A, Y)E + \eta gA\right)J \\
&\quad + e_1k\left(2g(\gamma A \times \gamma X) + \frac{1}{2}(\gamma A, X)E - \xi gA\right)J \text{ (Lemma 3.12.1)} \\
&= \chi\left(\begin{pmatrix} -2\gamma A \times Y + \eta A \\ 2A \times X - \xi \gamma A \\ (A, Y) \\ (-\gamma A, X) \end{pmatrix}\right) = \chi\left(\begin{pmatrix} 0 & -2\gamma A & 0 & A \\ 2A & 0 & -\gamma A & 0 \\ 0 & A & 0 & 0 \\ -\gamma A & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} X \\ Y \\ \xi \\ \eta \end{pmatrix}\right) \\
&= \chi(\Phi(0, A, -\gamma A, 0)P).
\end{aligned}$$

Hence we have $\varphi_*(e_1k(T)) = \Phi(0, A, -\gamma A, 0) \in (\mathfrak{e}_7)^{\tau\gamma}$.

Consequently we see that the mapping $\varphi : \mathfrak{su}(8) \rightarrow (\mathfrak{e}_7)^{\tau\gamma}$ is well-defined. We can easily check that φ_* is onto. Finally, we have to prove that φ_* is a homomorphism as Lie algebras. However this follows from the following Theorem 4.12.5, so that we shall omit the proof.

Lemma 4.12.4. *The group $(E_7)^{\tau\gamma}$ is connected.*

Proof. The group $(E_7)^{\tau\gamma}$ is the fixed points set by the involution $\tau\gamma$ of the simply connected Lie group E_7 , hence $(E_7)^{\tau\gamma}$ is connected.

Theorem 4.12.5. $(E_7)^{\tau\gamma} \cong SU(8)/\mathbf{Z}_2$, $\mathbf{Z}_2 = \{E, -E\}$.

Proof. We define a mapping $\varphi : SU(8) \rightarrow (E_7)^{\tau\gamma}$ by

$$\varphi(A)P = \chi^{-1}(A(\chi P)^t A), \quad P \in \mathfrak{P}^C.$$

We first prove $\varphi(A) \in (E_7)^{\tau\gamma}$. To prove this, for the differential mapping $\varphi_* : \mathfrak{su}(8) \rightarrow (\mathfrak{e}_7)^{\tau\gamma}$ of φ ,

$$\varphi_*(D)P = \chi^{-1}(D(\chi P) + (\chi P)^t D), \quad P \in \mathfrak{P}^C,$$

it is sufficient to show that φ_* is well-defined, that is, $\varphi_*(D) \in (\mathfrak{e}_7)^{\tau\gamma}$. However this fact is already shown in Proposition 4.12.3. Evidently $\varphi : SU(8) \rightarrow (E_7)^{\tau\gamma}$ is a homomorphism. Since $\varphi_* : \mathfrak{su}(8) \rightarrow (\mathfrak{e}_7)^{\tau\gamma}$ is onto and $(E_7)^{\tau\gamma}$ is connected (Lemma 4.12.4), $\varphi : SU(8) \rightarrow (E_7)^{\tau\gamma}$ is also onto. $\text{Ker } \varphi = \{E, -E\} = \mathbf{Z}_2$ is easily obtained. Thus we have the isomorphism $SU(8)/\mathbf{Z}_2 \cong (E_7)^{\tau\gamma}$.

Remark. Without using Lemma 4.12.4, the fact that the mapping $\varphi : SU(8) \rightarrow (E_7)^{\tau\gamma}$ is onto will be followed from two claims.

Claim 1. For $a \in \mathbf{R}$, $\alpha_i(a)$ of Lemma 4.8.1 belongs to $\varphi(SU(8))$.

Proof. $\alpha_i(a) = \exp(\Phi(0, -aE_i, aE_i, 0)) \in \exp \varphi_*(\mathfrak{su}(8))$ (Proposition 4.12.3) = $\varphi(\exp(\mathfrak{su}(8))) \in \varphi(SU(8))$.

Claim 2. Any element $P \in (\mathfrak{M}^C)_{\tau\gamma} = \{P \in \mathfrak{M}^C \mid \tau\gamma P = P\}$ can be transformed to a diagonal form by some $\alpha \in \varphi(SU(8))$:

$$\alpha P = (X, Y, \xi, \eta), \quad X, Y \text{ are real diagonal, } \xi > 0.$$

Proof. Let $P = (X, Y, \xi, \eta) \in (\mathfrak{M}^C)_{\tau\gamma}$. If $\xi \neq 0$. Then

$$\tau\gamma Y = Y, \quad X = \frac{1}{\xi}(Y \times Y), \quad \tau\xi = \xi, \quad \tau\eta = \eta.$$

Since $Y \in (\mathfrak{J}^C)_{\tau\gamma}$, we have $\gamma Y \in (\mathfrak{J}^C)_{\tau\gamma}$, so that $g(\gamma Y) \in \mathfrak{J}(4, \mathbf{H})_0$. Hence, there exists $D \in Sp(4)$ such that

$$D(g(\gamma Y))D^* \quad \text{is real diagonal.}$$

Then

$$\begin{aligned} \gamma\varphi(D)\gamma Y &= g^{-1}(D(g(\gamma Y))D^*) \text{ is real diagonal,} \\ \varphi(D)X &= \varphi(D)\left(\frac{1}{\xi}Y \times Y\right) = \frac{1}{\xi}(\gamma\varphi(D)\gamma Y \times \gamma\varphi(D)\gamma Y) \text{ is real diagonal.} \end{aligned}$$

In the case $\xi = 0$, by the same proof of Proposition 4.8.2, we can choose $\alpha \in \varphi(SU(8))$ so that

$$\alpha P = (X, Y, \xi, \eta), \quad X, Y \text{ are real diagonal, } 0 \neq \xi \in \mathbf{R}.$$

If $\xi < 0$, apply $\alpha_1(\pi)$ of Claim 1 on it, then ξ becomes $\xi > 0$.

Now, we will return to the proof of the surjection of $\varphi : SU(8) \rightarrow (E_7)^{\tau\gamma}$ using Claims 1, 2. For a given $\alpha \in (E_7)^{\tau\gamma}$, consider the element $P = \alpha \dot{1} \in (\mathfrak{M}^C)_{\tau\gamma}$. We first transform P to a diagonal form (Claim 2) by some $\beta \in \varphi(SU(8))$, and we have, in a similar way to Theorem 4.8.3,

$$\alpha_1(a_1)^{-1}\alpha_2(a_2)^{-1}\alpha_3(a_3)^{-1}\beta\alpha\dot{1} = \dot{1},$$

where $a_i = \frac{\eta_i}{|\eta_i|}r_i$ (η_i is a diagonal element of Y). Since $\eta_i \in \mathbf{R}$, we have $\alpha_i(a_i) \in \varphi(SU(8))$ (Claim 1). If we put $\tilde{\alpha} = \alpha_1(a_1)^{-1}\alpha_2(a_2)^{-1}\alpha_3(a_3)^{-1}\beta\alpha$, then, $\tilde{\alpha} \in E_6$ (Theorem 4.7.2) and $\tilde{\alpha}$ satisfies $\tau\gamma\tilde{\alpha} = \tilde{\alpha}\tau\gamma$. Hence $\tilde{\alpha} \in (E_6)^{\tau\gamma} = \varphi(Sp(4))$ (Theorem 3.12.2) $\subset \varphi(SU(8))$, Therefore $\alpha = \beta^{-1}\alpha_3(a_3)\alpha_2(a_2)\alpha_1(a_1)\tilde{\alpha} \in \varphi(SU(8))$. This shows that φ is onto.

4.13. Automorphism w of order 3 and subgroup $(SU(3) \times SU(6))/\mathbf{Z}_3$ of E_7

We define a C -linear transformation w of order 3 of \mathfrak{P}^C by

$$w(X, Y, \xi, \eta) = (wX, wY, \xi, \eta).$$

This w is the same as $w \in G_2 \subset F_4 \subset E_6 \subset E_7$.

We shall study the following subgroup $(E_7)^w$ of E_7 :

$$(E_7)^w = \{\alpha \in E_7 \mid w\alpha = \alpha w\}.$$

We consider the group $E_{7,C}$ replaced with C in the place \mathfrak{C} in the definition of the group E_7 :

$$E_{7,C} = \{\alpha \in \text{Iso}_C((\mathfrak{P}_C)^C) \mid \alpha(P \times Q)\alpha^{-1} = \alpha P \times \alpha Q, \langle \alpha P, \alpha Q \rangle = \langle P, Q \rangle\}.$$

As in Section 4.7, the group $E_{7,C}$ contains a subgroup

$$E_{6,C} = \{\alpha \in E_{7,C} \mid \alpha(0, 0, 1, 0) = (0, 0, 1, 0)\},$$

which is isomorphic to the group $((SU(3) \times SU(3))/\mathbf{Z}_3) \cdot \mathbf{Z}_2$ (Proposition 3.13.4). The Lie algebra $\mathfrak{e}_{7,C}$ of the group $E_{7,C}$ is given by

$$\mathfrak{e}_{7,C} = \{\Phi(\phi, A, -\tau A, \nu) \mid \phi \in \mathfrak{e}_{6,C}, A \in (\mathfrak{J}_C)^C, \nu \in i\mathbf{R}\}$$

(Theorem 4.3.4). In particular, the dimension of $\mathfrak{e}_{7,C}$ is

$$\dim \mathfrak{e}_{7,C} = 16 + 18 + 1 = 35.$$

As in Theorem 4.8.3, we see that the space

$$(\mathfrak{M}_C)_1 = \{P \in (\mathfrak{M}_C)^C \mid P \times P = 0, \langle P, P \rangle = 1\}$$

is connected and we have the homeomorphism

$$E_{7,C}/E_{6,C} \simeq (\mathfrak{M}_C)_1.$$

Lemma 4.13.1. *$E_{7,C}$ has at most two connected components (in reality has two connected components).*

Proof. From the exact sequence $\pi_0(E_{6,C}) \rightarrow \pi_0(E_{7,C}) \rightarrow \pi_0((\mathfrak{M}_C)_1)$, that is, $\mathbf{Z}_2 \rightarrow \pi_0(E_{7,C}) \rightarrow 0$ (Proposition 3.13.4), we see that $\pi_0(E_{7,C})$ is 0 or \mathbf{Z}_2 .

Let $h' : C \rightarrow C$ be the \mathbf{R} -linear isomorphism defined by

$$h'(a + bi) = a + be_1, \quad a, b \in \mathbf{R}.$$

Now, let V, W be C - and C -vector spaces, respectively. A linear mapping $f : V \rightarrow W$ is called a C - C -linear mapping if

$$f(av) = h'(a)f(v), \quad a \in C, v \in V.$$

Similarly, a C - C -linear mapping $g : W \rightarrow V$ is defined.

Definition. Let $h' : C^C \rightarrow C$ be a C - C -linear mapping defined by

$$h'(a + bi) = a + be_1, \quad a, b \in C.$$

Now, let $\Lambda^3(C^6)$ be the third exterior product of C -vector space C^6 and we define a C - C -linear isomorphism $f : (\mathfrak{P}_C)^C \rightarrow \Lambda^3(C^6)$ by

$$f\left(\begin{pmatrix} \xi_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & \xi_2 & x_1 \\ x_2 & \bar{x}_1 & \xi_3 \end{pmatrix}, \begin{pmatrix} \eta_1 & y_3 & \bar{y}_2 \\ \bar{y}_3 & \eta_2 & y_1 \\ y_2 & \bar{y}_1 & \eta_3 \end{pmatrix}, \xi, \eta\right) = \sum_{i < j < k} x_{ijk} \mathbf{e}_i \wedge \mathbf{e}_j \wedge \mathbf{e}_k$$

$\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_6\}$ is the canonical basis of C^6 and $x_{ijk} \in C$ are skew-symmetric tensor:

$$x_{i'j'k'} = \text{sgn} \begin{pmatrix} i & j & k \\ i' & j' & k' \end{pmatrix} x_{ijk}, \text{ where}$$

$$\begin{aligned}
x_{156} &= h'(\xi_1), & x_{164} &= h'(x_3), & x_{145} &= h'(\overline{x}_2), \\
x_{256} &= h'(\overline{x}_3), & x_{264} &= h'(\xi_2), & x_{245} &= h'(x_1), \\
x_{356} &= h'(x_2), & x_{364} &= h'(\overline{x}_1), & x_{345} &= h'(\xi_3), \\
x_{423} &= h'(\eta_1), & x_{431} &= h'(y_3), & x_{412} &= h'(\overline{y}_2), \\
x_{523} &= h'(\overline{y}_3), & x_{531} &= h'(\eta_2), & x_{512} &= h'(y_1), \\
x_{623} &= h'(y_2), & x_{631} &= h'(\overline{y}_1), & x_{612} &= h'(\eta_3), \\
x_{123} &= h'(\xi), \\
x_{456} &= h'(\eta).
\end{aligned}$$

The inverse mapping $f^{-1} : \Lambda^3(\mathbf{C}^6) \rightarrow (\mathfrak{P}_C)^C$ of f is given by

$$f^{-1}\left(\sum_{i<j<k} x_{ijk} \mathbf{e}_i \wedge \mathbf{e}_j \wedge \mathbf{e}_k\right) = \begin{pmatrix} \begin{pmatrix} h(x_{156}) & h(x_{164}, \overline{x}_{256}) & h(x_{145}, \overline{x}_{356}) \\ h(x_{256}, \overline{x}_{164}) & h(x_{264}) & h(x_{245}, \overline{x}_{364}) \\ h(x_{356}, \overline{x}_{145}) & h(x_{364}, \overline{x}_{245}) & h(x_{345}) \end{pmatrix} \\ \begin{pmatrix} h(x_{423}) & h(x_{431}, \overline{x}_{523}) & h(x_{412}, \overline{x}_{623}) \\ h(x_{523}, \overline{x}_{431}) & h(x_{531}) & h(x_{512}, \overline{x}_{631}) \\ h(x_{623}, \overline{x}_{412}) & h(x_{631}, \overline{x}_{512}) & h(x_{612}) \end{pmatrix} \\ h(x_{123}) \\ h(x_{456}) \end{pmatrix},$$

where $h : \mathbf{C} \oplus \mathbf{C} \rightarrow \mathbf{C}^C, h : \mathbf{C} \rightarrow \mathbf{C}$ are \mathbf{C} - \mathbf{C} -linear mappings defined respectively by

$$\begin{aligned}
h(a, b) &= \frac{a+b}{2} + i \frac{(b-a)e_1}{2}, \quad a, b \in \mathbf{C}, \\
h(a + be_1) &= a + bi, \quad a, b \in \mathbf{R}.
\end{aligned}$$

It is easy to see that

$$f(h(a)P) = a(fP), \quad a \in \mathbf{C}, P \in (\mathfrak{P}_C)^C.$$

The group $SU(6)$ acts naturally on $\Lambda^3(\mathbf{C}^6)$, that is, the action of $A \in SU(6)$ on $\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c} \in \Lambda^3(\mathbf{C}^6)$ is defined by

$$A(\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}) = A\mathbf{a} \wedge A\mathbf{b} \wedge A\mathbf{c}.$$

Hence, the action of $D \in \mathfrak{su}(6)$ on $\Lambda^3(\mathbf{C}^6)$ is given by

$$D(\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}) = D\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c} + \mathbf{a} \wedge D\mathbf{b} \wedge \mathbf{c} + \mathbf{a} \wedge \mathbf{b} \wedge D\mathbf{c}.$$

Lemma 4.13.2. (1) Any element $D \in \mathfrak{su}(6)$ is uniquely expressed by

$$D = \begin{pmatrix} B & L \\ -L^* & C \end{pmatrix} + \frac{\nu}{3} \begin{pmatrix} E & 0 \\ 0 & -E \end{pmatrix}, \quad B, C \in \mathfrak{su}(3), L \in M(3, \mathbf{C}), \nu \in e_1 \mathbf{R}.$$

(2) The Lie algebra $\mathfrak{e}_{7,C}$ is isomorphic to the Lie algebra $\mathfrak{su}(6)$ as Lie algebras:

$$\mathfrak{e}_{7,C} \cong \mathfrak{su}(6).$$

This isomorphism is given by the mapping $\varphi_C : \mathfrak{su}(6) \rightarrow \mathfrak{e}_{7,C}$,

$$\varphi_C \left(\begin{pmatrix} B & L \\ -L^* & C \end{pmatrix} + \frac{\nu}{3} \begin{pmatrix} E & 0 \\ 0 & -E \end{pmatrix} \right) = \Phi(\phi_C(B, C), h(L), -\tau h(L), -i\nu e_1)$$

where $\phi_C : \mathfrak{su}(3) \oplus \mathfrak{su}(3) \rightarrow \mathfrak{e}_{6,C}$ is defined by $\phi_C(B, C)X = h(B, C)X + Xh(B, C)^*$, $X \in (\mathfrak{J}_C)^C$ (Lemma 3.13.3).

Proof. (1) For $D = \begin{pmatrix} B' & L \\ -L^* & C' \end{pmatrix} \in \mathfrak{su}(6)$, $B', C' \in \mathfrak{u}(3)$, $L \in M(3, C)$, we let

$$\nu = \text{tr}(B') = -\text{tr}(C'), \quad B = B' - \frac{\nu}{3}E, \quad C = C' + \frac{\nu}{3}E,$$

then we have the result.

(2) This is the direct consequence of the following Proposition 4.13.3, so we will omit its proof.

We define the action of the group $Z_2 = \{1, \epsilon\}$ on the group $SU(6)$ by

$$\epsilon A = \overline{(\text{Ad} J_3)A}, \quad J_3 = \begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix},$$

that is,

$$\epsilon A = \epsilon \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \overline{\begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix}^{-1}} = \begin{pmatrix} \overline{A_{22}} & -\overline{A_{21}} \\ -\overline{A_{12}} & \overline{A_{11}} \end{pmatrix},$$

where $E, A_{ij} \in M(3, C)$, and let $SU(6) \cdot Z_2$ be the semi-direct product of the groups $SU(6)$ and Z_2 under this action.

Proposition 4.13.3. $E_{7,C} \cong (SU(6)/Z_3) \cdot Z_2$, $Z_3 = \{E, \omega_1 E, \omega_1^2 E\}$, $\omega_1 = -\frac{1}{2} + \frac{\sqrt{3}}{2}e_1$.

Proof. We define a mapping $\psi : SU(6) \cdot Z_2 \rightarrow E_{7,C}$ by

$$\psi(A, 1)P = f^{-1}(A(fP)), \quad \psi(A, \epsilon)P = f^{-1}(A(f\overline{P})), \quad P \in (\mathfrak{P}_C)^C.$$

We first have to show that $\psi(A, 1) \in E_{7,C}$. To prove this, it is sufficient to show that the differential mapping $\psi_* : \mathfrak{su}(6) \rightarrow \mathfrak{e}_{7,C}$ of ψ :

$$\psi_*(D)P = f^{-1}(D(fP)), \quad P \in (\mathfrak{P}_C)^C$$

coincides with the mapping $\psi_C : \mathfrak{su}(6) \rightarrow \mathfrak{e}_{7,C}$ of Lemma 4.13.2. We put

$$D = \begin{pmatrix} B & L \\ -L^* & C \end{pmatrix} + \frac{\nu}{3} \begin{pmatrix} E & 0 \\ 0 & -E \end{pmatrix}$$

$$= \begin{pmatrix} b_{11} & b_{12} & b_{13} & l_{11} & l_{12} & l_{13} \\ -\bar{b}_{12} & b_{22} & b_{23} & l_{21} & l_{22} & l_{23} \\ -\bar{b}_{13} & b_{23} & b_{33} & l_{31} & l_{32} & l_{33} \\ -\bar{l}_{11} & -\bar{l}_{21} & -\bar{l}_{31} & c_{11} & c_{12} & c_{13} \\ -\bar{l}_{12} & -\bar{l}_{22} & -\bar{l}_{32} & -\bar{c}_{12} & c_{22} & c_{23} \\ -\bar{l}_{13} & -\bar{l}_{23} & -\bar{l}_{33} & -\bar{c}_{13} & -\bar{c}_{23} & c_{33} \end{pmatrix} + \frac{\nu}{3} \begin{pmatrix} E & 0 \\ 0 & -E \end{pmatrix} \in \mathfrak{su}(6),$$

where $\bar{b}_{ii} = -b_{ii}$, $\bar{c}_{ii} = -c_{ii}$, $b_{11} + b_{22} + b_{33} = c_{11} + c_{22} + c_{33} = 0$, $\bar{\nu} = -\nu$.

(1) For $P = (0, 0, 1, 0)$, $\psi_*(D)P$ is calculated as follows.

$$\begin{aligned} P &= (0, 0, 1, 0) \\ &\xrightarrow{f} \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3 \\ &\xrightarrow{D} D\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3 + \mathbf{e}_1 \wedge D\mathbf{e}_2 \wedge \mathbf{e}_3 + \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge D\mathbf{e}_3 \\ &= \left(b_{11} + \frac{\nu}{3}\right) \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3 - \bar{l}_{11} \mathbf{e}_4 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3 - \bar{l}_{12} \mathbf{e}_5 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3 - \bar{l}_{13} \mathbf{e}_6 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3 \\ &\quad + \left(b_{22} + \frac{\nu}{3}\right) \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3 - \bar{l}_{21} \mathbf{e}_4 \wedge \mathbf{e}_3 \wedge \mathbf{e}_1 - \bar{l}_{22} \mathbf{e}_1 \wedge \mathbf{e}_5 \wedge \mathbf{e}_3 - \bar{l}_{23} \mathbf{e}_1 \wedge \mathbf{e}_6 \wedge \mathbf{e}_3 \\ &\quad + \left(b_{33} + \frac{\nu}{3}\right) \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3 - \bar{l}_{31} \mathbf{e}_4 \wedge \mathbf{e}_1 \wedge \mathbf{e}_2 - \bar{l}_{32} \mathbf{e}_5 \wedge \mathbf{e}_1 \wedge \mathbf{e}_2 - \bar{l}_{33} \mathbf{e}_6 \wedge \mathbf{e}_1 \wedge \mathbf{e}_2 \\ &\xrightarrow{f^{-1}} \begin{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ \begin{pmatrix} -h(\bar{l}_{11}) & -h(\bar{l}_{21}, l_{12}) & -h(\bar{l}_{31}, l_{13}) \\ -h(\bar{l}_{12}, l_{21}) & -h(\bar{l}_{22}) & -h(\bar{l}_{32}, l_{23}) \\ -h(\bar{l}_{13}, l_{31}) & -h(\bar{l}_{23}, l_{32}) & -h(\bar{l}_{33}) \end{pmatrix} \\ h(\nu) \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -\tau h(L) \\ -ive_1 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} \phi_C(B, C) + \frac{ive_1}{3} & -2\tau h(L) & 0 & h(L) \\ 2h(L) & \tau\phi_C(B, C)\tau - \frac{ive_1}{3} & -\tau h(L) & 0 \\ 0 & h(L) & -ive_1 & 0 \\ -\tau h(L) & 0 & 0 & ive_1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \\ &= \Phi(\phi_C(B, C), h(L), -\tau h(L), -ive_1)P. \end{aligned}$$

(2) For $P = (E_1, 0, 0, 0)$, $\psi_*(D)P$ is calculated as follows.

$$\begin{aligned} P &= (E_1, 0, 0, 0) \\ &\xrightarrow{f} \mathbf{e}_1 \wedge \mathbf{e}_5 \wedge \mathbf{e}_6 \\ &\xrightarrow{D} D\mathbf{e}_1 \wedge \mathbf{e}_5 \wedge \mathbf{e}_6 + \mathbf{e}_1 \wedge D\mathbf{e}_5 \wedge \mathbf{e}_6 + \mathbf{e}_1 \wedge \mathbf{e}_5 \wedge D\mathbf{e}_6 \\ &= \left(b_{11} + \frac{\nu}{3}\right) \mathbf{e}_1 \wedge \mathbf{e}_5 \wedge \mathbf{e}_6 - \bar{b}_{12} \mathbf{e}_2 \wedge \mathbf{e}_5 \wedge \mathbf{e}_6 - \bar{b}_{13} \mathbf{e}_3 \wedge \mathbf{e}_5 \wedge \mathbf{e}_6 - \bar{l}_{11} \mathbf{e}_4 \wedge \mathbf{e}_5 \wedge \mathbf{e}_6 \\ &\quad + l_{22} \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_6 + l_{32} \mathbf{e}_1 \wedge \mathbf{e}_3 \wedge \mathbf{e}_6 + c_{12} \mathbf{e}_1 \wedge \mathbf{e}_4 \wedge \mathbf{e}_6 + \left(c_{22} - \frac{\nu}{3}\right) \mathbf{e}_1 \wedge \mathbf{e}_5 \wedge \mathbf{e}_6 \\ &\quad + l_{23} \mathbf{e}_1 \wedge \mathbf{e}_5 \wedge \mathbf{e}_2 + l_{33} \mathbf{e}_1 \wedge \mathbf{e}_5 \wedge \mathbf{e}_3 + c_{13} \mathbf{e}_1 \wedge \mathbf{e}_5 \wedge \mathbf{e}_4 + \left(c_{33} - \frac{\nu}{3}\right) \mathbf{e}_1 \wedge \mathbf{e}_5 \wedge \mathbf{e}_6 \end{aligned}$$

$$\begin{aligned}
& \xrightarrow{f^{-1}} \begin{pmatrix} \begin{pmatrix} h(b_{11} - c_{11} - \frac{\nu}{3}) & -h(c_{12}, b_{12}) & -h(c_{13}, b_{13}) \\ -h(\bar{b}_{12}, \bar{c}_{12}) & 0 & 0 \\ -h(\bar{b}_{13}, \bar{c}_{13}) & 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 & 0 \\ 0 & h(l_{33}) & -h(l_{23}, \bar{l}_{32}) \\ 0 & -h(l_{32}, \bar{l}_{23}) & h(l_{22}) \end{pmatrix} \\ 0 \\ -h(\bar{l}_{11}) \end{pmatrix} \\
& = \begin{pmatrix} \phi_{\mathbf{C}}(B, C)E_1 + \frac{i\nu e_1}{3}E_1 \\ 2h(L) \times E_1 \\ 0 \\ -(\tau h(L), E_1) \end{pmatrix} = \Phi(\phi_{\mathbf{C}}(B, C), h(L), -\tau h(L), -i\nu e_1) \begin{pmatrix} E_1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.
\end{aligned}$$

(3) For $P = (F_1(1), 0, 0, 0)$, $\psi_*(D)P$ is calculated as follows.

$$P = (F_1(1), 0, 0, 0)$$

$$\begin{aligned}
& \xrightarrow{f} \mathbf{e}_2 \wedge \mathbf{e}_4 \wedge \mathbf{e}_5 + \mathbf{e}_3 \wedge \mathbf{e}_6 \wedge \mathbf{e}_4 \\
& \xrightarrow{D} (D\mathbf{e}_2 \wedge \mathbf{e}_4 \wedge \mathbf{e}_5 + \mathbf{e}_2 \wedge D\mathbf{e}_4 \wedge \mathbf{e}_5 + \mathbf{e}_2 \wedge \mathbf{e}_4 \wedge D\mathbf{e}_5) \\
& \quad + (D\mathbf{e}_3 \wedge \mathbf{e}_6 \wedge \mathbf{e}_4 + \mathbf{e}_3 \wedge D\mathbf{e}_6 \wedge \mathbf{e}_4 + \mathbf{e}_3 \wedge \mathbf{e}_6 \wedge D\mathbf{e}_4) \\
& = \left(b_{12}\mathbf{e}_1 \wedge \mathbf{e}_4 \wedge \mathbf{e}_5 + \left(b_{22} + \frac{\nu}{3} \right) \mathbf{e}_2 \wedge \mathbf{e}_4 \wedge \mathbf{e}_5 - \bar{b}_{23}\mathbf{e}_3 \wedge \mathbf{e}_4 \wedge \mathbf{e}_5 - \bar{l}_{23}\mathbf{e}_6 \wedge \mathbf{e}_4 \wedge \mathbf{e}_5 \right. \\
& \quad + l_{11}\mathbf{e}_2 \wedge \mathbf{e}_1 \wedge \mathbf{e}_5 + l_{31}\mathbf{e}_2 \wedge \mathbf{e}_3 \wedge \mathbf{e}_5 + \left(c_{11} - \frac{\nu}{3} \right) \mathbf{e}_2 \wedge \mathbf{e}_4 \wedge \mathbf{e}_5 - \bar{c}_{13}\mathbf{e}_2 \wedge \mathbf{e}_6 \wedge \mathbf{e}_5 \\
& \quad + l_{12}\mathbf{e}_2 \wedge \mathbf{e}_4 \wedge \mathbf{e}_1 + l_{32}\mathbf{e}_2 \wedge \mathbf{e}_4 \wedge \mathbf{e}_3 + \left(c_{22} - \frac{\nu}{3} \right) \mathbf{e}_3 \wedge \mathbf{e}_6 \wedge \mathbf{e}_4 - \bar{c}_{23}\mathbf{e}_2 \wedge \mathbf{e}_4 \wedge \mathbf{e}_6 \Big) \\
& \quad + \left(b_{13}\mathbf{e}_1 \wedge \mathbf{e}_6 \wedge \mathbf{e}_4 + b_{23}\mathbf{e}_2 \wedge \mathbf{e}_6 \wedge \mathbf{e}_4 + \left(b_{33} + \frac{\nu}{3} \right) \mathbf{e}_3 \wedge \mathbf{e}_6 \wedge \mathbf{e}_4 - \bar{l}_{32}\mathbf{e}_5 \wedge \mathbf{e}_6 \wedge \mathbf{e}_4 \right. \\
& \quad + l_{13}\mathbf{e}_3 \wedge \mathbf{e}_1 \wedge \mathbf{e}_4 + l_{23}\mathbf{e}_3 \wedge \mathbf{e}_2 \wedge \mathbf{e}_4 + c_{23}\mathbf{e}_3 \wedge \mathbf{e}_5 \wedge \mathbf{e}_4 + \left(c_{33} - \frac{\nu}{3} \right) \mathbf{e}_3 \wedge \mathbf{e}_6 \wedge \mathbf{e}_4 \\
& \quad \left. + l_{11}\mathbf{e}_3 \wedge \mathbf{e}_6 \wedge \mathbf{e}_1 + l_{21}\mathbf{e}_3 \wedge \mathbf{e}_6 \wedge \mathbf{e}_2 + \left(c_{11} - \frac{\nu}{3} \right) \mathbf{e}_3 \wedge \mathbf{e}_6 \wedge \mathbf{e}_4 - \bar{c}_{12}\mathbf{e}_3 \wedge \mathbf{e}_6 \wedge \mathbf{e}_5 \right) \\
& \xrightarrow{f^{-1}} \begin{pmatrix} \begin{pmatrix} 0 & h(b_{13}, c_{13}) & 0 \\ * & h(b_{23} + \bar{c}_{23}) & h(b_{22} - c_{33} - \frac{\nu}{3}, -b_{33} + c_{33} + \frac{\nu}{3}) \\ h(\bar{c}_{12}, \bar{b}_{12}) & * & -h(\bar{b}_{23} + c_{23}) \end{pmatrix} \\ \begin{pmatrix} -h(l_{23} + l_{32}) & h(l_{13}, \bar{l}_{31}) & * \\ * & 0 & -h(l_{11}) \\ h(l_{21}, \bar{l}_{12}) & * & 0 \end{pmatrix} \\ 0 \\ -h(\bar{l}_{23} + \bar{l}_{32}) \end{pmatrix}
\end{aligned}$$

$$= \begin{pmatrix} \phi_{\mathbf{C}}(B, C)F_1(1) + \frac{i\nu e_1}{3}F_1(1) \\ 2h(L) \times F_1(1) \\ 0 \\ -(h(L), F_1(1)) \end{pmatrix} = \Phi(\phi_{\mathbf{C}}(B, C), h(L), -\tau h(L), -i\nu e_1) \begin{pmatrix} F_1(1) \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

(4) For the other generator of $\mathfrak{P}_{\mathbf{C}}^C$, that is, $P = (X, 0, 0, 0), (0, X, 0, 0)$ where $X = E_i, F_i(1), F_i(e_1), i = 1, 2, 3$ and $P = (0, 0, 0, 1)$, we have also

$$f^{-1}(D(fP)) = \Phi(\phi_{\mathbf{C}}(B, C), h(L), -\tau h(L), -i\nu e_1)P.$$

Thus we see that $\psi(A, 1) \in E_{7, \mathbf{C}}$ for $A \in SU(6)$. Since $\psi(E, \epsilon)P = \overline{P}$, we see $\psi(E, \epsilon) = \epsilon \in G_{2, \mathbf{C}} (= \text{Aut}(\mathbf{C})) \subset F_{4, \mathbf{C}} \subset E_{6, \mathbf{C}} \subset E_{7, \mathbf{C}}$. We shall show that $\psi : SU(6) \cdot \mathbf{Z}_2 \rightarrow E_{7, \mathbf{C}}$ is a homomorphism. For this purpose, we first show

$$\overline{f^{-1}(A(fP))} = f^{-1}(\overline{((\text{Ad} J_3)A f \overline{P})}), \quad A \in SU(6), P \in (\mathfrak{P}_{\mathbf{C}})^C.$$

Furthermore, to show this, it is sufficient to show that for $D \in \mathfrak{su}(6)$ instead of $A \in SU(6)$. Now,

$$\begin{aligned} \overline{f^{-1}(D(fP))} &= \overline{\Phi(\phi_{\mathbf{C}}(B, C), h(L), -\tau h(L), -i\nu e_1)P} \\ &= \Phi(\overline{\phi_{\mathbf{C}}(B, C)}, \overline{h(L)}, -\tau \overline{h(L)}, -i\nu e_1)\overline{P} \\ &= \Phi(\phi_{\mathbf{C}}(\overline{C}, \overline{B}), h(\overline{L}^*), -\tau h(\overline{L}^*), -i\nu e_1)\overline{P} \\ &= f^{-1}(\overline{((\text{Ad} J_3)D)(f\overline{P})}). \end{aligned}$$

ψ is a homomorphism. Indeed, for example,

$$\begin{aligned} \psi(A, \epsilon)\psi(B, 1)P &= \psi(A, \epsilon)(f^{-1}(B(fP))) \\ &= f^{-1}(Af(\overline{f^{-1}(B(fP))})) = f^{-1}(Af(f^{-1}(\overline{((\text{Ad} J_3)B)(f\overline{P})}))) \\ &= f^{-1}((A(\epsilon B))(f\overline{P})) = \psi(A(\epsilon B), \epsilon)P. \end{aligned}$$

Thus, for $A \in SU(6)$, we have $\psi(A, \epsilon) = \varphi(A, 1)\varphi(E, \epsilon) \in E_{7, \mathbf{C}}$. Since ψ induces a surjection $\psi_* : \mathfrak{su}(6) \rightarrow \mathfrak{e}_{7, \mathbf{C}}$, $\psi : SU(6) \rightarrow (E_{7, \mathbf{C}})_0$ (which is the connected component of $E_{7, \mathbf{C}}$ containing the identity 1) is onto. However $\epsilon = \psi(E, \epsilon) \notin (E_{7, \mathbf{C}})_0$. Indeed, for any $A \in SU(6)$,

$$\sum (Aa \wedge Ab \wedge Ac) = \overline{a} \wedge \overline{b} \wedge \overline{c}, \quad a, b, c \in \mathbf{C}^6$$

does not hold. Therefore $E_{7, \mathbf{C}}$ has just two connected components (see Lemma 4.13.1). Hence $\psi : SU(6) \cdot \mathbf{Z}_2 \rightarrow E_{7, \mathbf{C}}$ is onto. $\text{Ker } \psi = \{E, \omega_1 E, \omega_1^2 E\} \times 1 = \mathbf{Z}_3 \times 1$ easily obtained. Thus we have the isomorphism $(SU(6)/\mathbf{Z}_3) \cdot \mathbf{Z}_2 \cong E_{7, \mathbf{C}}$.

We identify $(\mathfrak{P}_{\mathbf{C}})^C \oplus (M(3, \mathbf{C})^C \oplus M(3, \mathbf{C})^C)$ with \mathfrak{P}^C (using the identification $\mathfrak{J}_{\mathbf{C}}^C \oplus M(3, \mathbf{C})^C$ with \mathfrak{J}^C in Section 3.13) by

$$((X, Y, \xi, \eta), (M, N)) = (X + M, Y + N, \xi, \eta).$$

Further, we define a \mathbf{C} - \mathbf{C} -linear mapping $\mu : M(6, \mathbf{C}) \rightarrow M(3, \mathbf{C})^{\mathbf{C}} \oplus M(3, \mathbf{C})^{\mathbf{C}}$ by

$$\begin{aligned} \mu \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \\ = \left(\frac{(M_{21} - M_{12})e_1}{2} + i \frac{M_{21} + M_{12}}{2}, \frac{(M_{22} + M_{11})e_1}{2} + i \frac{M_{22} - M_{11}}{2} \right), \end{aligned}$$

where $M_{ij} \in M(3, \mathbf{C})$. The inverse mapping $\mu^{-1} : M(3, \mathbf{C})^{\mathbf{C}} \oplus M(3, \mathbf{C})^{\mathbf{C}} \rightarrow M(6, \mathbf{C})$ of μ is given by

$$\mu^{-1}(M_1 + iM_2, N_1 + iN_2) = \begin{pmatrix} -N_2 - N_1e_1 & M_2 + M_1e_1 \\ M_2 - M_1e_1 & N_2 - N_1e_1 \end{pmatrix}, \quad M_i, N_i \in M(3, \mathbf{C}).$$

Lemma 4.13.4. *For $D \in \mathfrak{su}(6)$ and $\widetilde{M} \in M(6, \mathbf{C})$, we have*

$$\mu(\widetilde{M}D^*) = \psi_*(D)(\mu\widetilde{M}).$$

Proof. Let

$$\begin{aligned} D &= \begin{pmatrix} B & L \\ -L^* & C \end{pmatrix} + \frac{\nu}{3} \begin{pmatrix} E & 0 \\ 0 & -E \end{pmatrix} \in \mathfrak{su}(6), \\ \widetilde{M} &= \begin{pmatrix} -N_2 - N_1e_1 & M_2 + M_1e_1 \\ M_2 - M_1e_1 & N_2 - N_1e_1 \end{pmatrix}, \quad M_i, N_i \in M(3, \mathbf{C}), \\ M &= M_1 + iM_2, N = N_1 + iN_2. \end{aligned}$$

Then we have

$$\begin{aligned} &\psi_*(D)(\mu\widetilde{M}) \\ &= \Phi(\phi_{\mathbf{C}}(B, C), h(L), -\tau h(L), -ive_1)(M, N) \\ &= \begin{pmatrix} \phi_{\mathbf{C}}(B, C) + \frac{1}{3}ive_1 & -2\tau h(L) & 0 & h(L) \\ 2h(L) & \tau\phi_{\mathbf{C}}(B, C)\tau - \frac{1}{3}ive_1 & -\tau h(L) & 0 \\ 0 & h(L) & -ive_1 & 0 \\ -\tau h(L) & 0 & 0 & ive_1 \end{pmatrix} \begin{pmatrix} M \\ N \\ 0 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} \phi_{\mathbf{C}}(B, C)M + \frac{1}{3}ive_1M - 2\tau h(L) \times N \\ 2h(L) \times M + \tau\phi_{\mathbf{C}}(B, C)\tau N - \frac{1}{3}ive_1N \\ (h(L), N) \\ -(\tau h(L), M) \end{pmatrix} \\ &= \begin{pmatrix} -Mh(B, C) + N\tau h(L) + \frac{1}{3}ive_1M \\ -Mh(L) - N\tau h(B, C) - \frac{1}{3}ive_1N \\ 0 \\ 0 \end{pmatrix} \end{aligned}$$

(using $\phi_C(B, C)M = M\tau h(B, C)^* = -Mh(B, C)$ and $-2\tau h(L) \times N = N\tau h(L)$ etc.)

$$\begin{aligned} & \xrightarrow{\mu^{-1}} \dots \text{ by simple calculations } \dots \\ & = \begin{pmatrix} -N_2 - N_1 e_1 & M_2 + M_1 e_1 \\ M_2 + M_1 e_1 & N_2 + N_1 e_1 \end{pmatrix} \left(\begin{pmatrix} -B & -L \\ L^* & -C \end{pmatrix} - \frac{\nu}{3} \begin{pmatrix} E & 0 \\ 0 & -E \end{pmatrix} \right) = \widetilde{M}D^*. \end{aligned}$$

Definition. We define a C - C -linear isomorphism $f : \mathfrak{P}^C \rightarrow \Lambda^3(C^6) \oplus M(6, C)$ by

$$\begin{aligned} f(P_C + (M + N)) &= f(P_C) + \mu^{-1}(M + N), \\ P_C + (M + N) &\in (\mathfrak{P}_C)^C \oplus (M(3, C)^C \oplus M(3, C)^C) = \mathfrak{P}^C. \end{aligned}$$

The group $SU(3) \times SU(6)$ acts on $\Lambda^3(C^6) \oplus M(6, C)$ by

$$(Q, A)(\sum(\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}) + \widetilde{M}) = \sum(A\mathbf{a} \wedge A\mathbf{b} \wedge A\mathbf{c}) + Q\widetilde{M}A^*,$$

where $Q\widetilde{M}$ means $\begin{pmatrix} Q & 0 \\ 0 & Q \end{pmatrix} \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} = \begin{pmatrix} QM_{11} & QM_{12} \\ QM_{21} & QM_{22} \end{pmatrix}$, $M_{ij} \in M(3, C)$.

Theorem 4.13.5. $(E_7)^w \cong (SU(3) \times SU(6))/\mathbf{Z}_3$, $\mathbf{Z}_3 = \{(E, E), (\omega_1 E, \omega_1 E), (\omega_1^2 E, \omega_1^2 E)\}$, $\omega_1 = -\frac{1}{2} + \frac{\sqrt{3}}{2}e_1$.

Proof. We defined a mapping $\psi : SU(3) \times SU(6) \rightarrow (E_7)^w$ by

$$\psi(Q, A)P = f^{-1}((Q, A)(fP)), \quad P \in \mathfrak{P}^C.$$

We first have to prove that $\psi(Q, A) \in E_7$. To prove this, since $\varphi(Q, E) \in (E_6)^w \subset (E_7)^w$, it suffices to show that $\psi(E, A) \in E_7$. Moreover, it is sufficient to show that, for the differential mapping $\psi_* : \mathfrak{su}(3) \oplus \mathfrak{su}(6) \rightarrow \mathfrak{e}_7$ of ψ , $\psi_*(0, D)$ coincides with

$$\Phi(\phi_C(B, C), h(L), -\tau h(L), -ive_1) \in \mathfrak{e}_7.$$

However, this is already shown in Proposition 4.13.3 and Lemma 4.13.4. Since $(\mathfrak{P}^C)_w = \{P \in \mathfrak{P}^C \mid wP = P\} = (\mathfrak{P}_C)^C$, obviously we have $w\psi(Q, A) = \psi(Q, A)w$, hence $\psi(Q, A) \in (E_7)^w$. Evidently ψ is a homomorphism. We shall show that ψ is onto. Let $\alpha \in (E_7)^w$. Since the restriction α' to $(\mathfrak{P}^C)_w = (\mathfrak{P}_C)^C$ of α belongs to $E_{7,C}$, there exists $A \in SU(6)$ such that

$$\alpha P = f^{-1}(A(fP)) \quad \text{or} \quad \alpha P = f^{-1}(A(f\overline{P})), \quad P \in (\mathfrak{P}_C)^C$$

(Proposition 4.13.3). In the former case, let $\beta = \psi(E, A)^{-1}\alpha$, then $\beta|(\mathfrak{P}_C)^C = 1$, hence $\beta \in G_2$. Furthermore, $\beta \in (G_2)^w = SU(3)$ (Theorem 1.9.4), so there exists $Q \in SU(3)$ such that

$$\begin{aligned} \beta(P_C + (M + N)) &= P_C + Q(M + N) = P_C + (QM + QN) \\ &= \psi(Q, E)(P_C + (M + N)), \quad P_C + (M + N) \in \mathfrak{P}^C \end{aligned}$$

Hence we have

$$\alpha = \psi(E, A)\beta = \psi(E, A)\psi(Q, E) = \psi(Q, A).$$

In this case, this shows that ψ is onto. In the latter case, consider the mapping

$$\gamma_1 : \mathfrak{P}^C \rightarrow \mathfrak{P}^C, \gamma_1(P_C + (M + N)) = \overline{P_C} + (\overline{M} + \overline{N}), \quad P_C + (M + N) \in \mathfrak{P}^C.$$

Then, $\gamma_1 \in G_2 \subset F_4 \subset E_6 \subset E_7$. From the same argument as Section 1, we have $\gamma_1 \alpha \in (E_7)^w$, hence $\gamma_1 \in (G_2)^w = SU(3)$. However this is a contradiction (Theorem 1.9.4). Therefore that ψ is onto is shown. $\text{Ker}\psi = \{(E, E), (\omega_1 E, \omega_1 E), (\omega_1^2 E, \omega_1^2 E)\} = \mathbf{Z}_3$ is easily obtained. Thus we have the isomorphism $(SU(3) \times SU(6))/\mathbf{Z}_3 \cong (E_7)^w$.

Remark 1. The group E_7 has a subgroup which is isomorphic to the semi-direct product $((SU(3) \times SU(6))/\mathbf{Z}_3) \cdot \mathbf{Z}_2$ (the action of the group $\mathbf{Z}_2 = \{1, \gamma\}$ to the group $SU(3) \times SU(6)$ is $\gamma(Q, A) = (\overline{Q}, \text{Ad}(J_3)A)$).

Remark 2. Since $(E_7)^w$ is connected, the fact that $\psi : SU(3) \times SU(6) \rightarrow (E_7)^w$ is onto can be proved as follows. The elements

$$\begin{aligned} &G_{01}, \quad G_{23}, \quad G_{45}, \quad G_{67}, \quad G_{46} + G_{47}, \quad G_{47} - G_{56}, \\ &G_{24} + G_{35}, \quad G_{25} - G_{34}, \quad G_{26} + G_{37}, \quad G_{27} - G_{36}, \\ &\tilde{A}_l(1), \quad \tilde{A}_l(e_1), \quad \tilde{F}_l(1), \quad \tilde{F}_l(e_1), \quad (E_1 - E_2)^\sim, \quad (E_2 - E_3)^\sim \\ &\check{F}_l(1), \quad \check{F}_l(e_1), \quad \hat{F}_l(1), \quad \hat{F}_l(e_1), \quad \check{E}_l, \quad \hat{E}_l, \quad 1, \quad l = 1, 2, 3 \end{aligned}$$

forms an \mathbf{R} -basis of $(\mathfrak{e}_7)^w$. So, $\dim(\mathfrak{e}_7)^w = 10 + 14 + 6 \times 3 + 1 = 43 = 8 + 35 = \dim(\mathfrak{su}(3) \oplus \mathfrak{su}(6))$. Hence φ is onto.

4.14. Complex exceptional Lie group E_7^C

Theorem 4.14.1. *The polar decomposition of the Lie group E_7^C is given by*

$$E_7^C \simeq E_7 \times \mathbf{R}^{133}.$$

In particular, E_7^C is a simply connected complex Lie group of type E_7 .

Proof. Evidently E_7^C is an algebraic subgroup of $\text{Iso}_C(\mathfrak{P}^C) = GL(78, C)$. If $\alpha \in E_7^C$, then, the complex conjugate transpose α^* with respect to the inner product $\langle X, Y \rangle$: $\langle \alpha X, Y \rangle = \langle X, \alpha^* Y \rangle$ is $\alpha^* = \tau \lambda \alpha^{-1} \lambda^{-1} \tau \in E_7^C$. Hence, from Chevalley's lemma, we have

$$E_7^C \simeq (E_7^C \cap U(\mathfrak{P}^C)) \times \mathbf{R}^d = E_7 \times \mathbf{R}^d, \quad d = 133.$$

Since E_7 is simply connected (Theorem 4.9.2), E_7^C is also simply connected. The Lie algebra of the group E_7^C is \mathfrak{e}_7^C , so E_7^C is a complex simple Lie group of type E_7 .

4.15. Non-compact exceptional Lie groups $E_{7(7)}, E_{7(-5)}$ and $E_{7(-25)}$ of type E_7

Let

$$\begin{aligned}\mathfrak{P} &= \mathfrak{J}(3, \mathfrak{C}) \oplus \mathfrak{J}(3, \mathfrak{C}) \oplus \mathbf{R} \oplus \mathbf{R}, \\ \mathfrak{P}' &= \mathfrak{J}(3, \mathfrak{C}') \oplus \mathfrak{J}(3, \mathfrak{C}') \oplus \mathbf{R} \oplus \mathbf{R}.\end{aligned}$$

For $P, Q \in \mathfrak{P}$ or \mathfrak{P}' , we define an \mathbf{R} -linear mapping $P \times Q : \mathfrak{P} \rightarrow \mathfrak{P}$ or $\mathfrak{P}' \rightarrow \mathfrak{P}'$ as similar to Section 4.1. And we define a Hermitian inner product $\langle P, Q \rangle_\sigma$ in \mathfrak{P}^C by

$$\langle P, Q \rangle_\sigma = \langle \sigma P, Q \rangle.$$

Now, we define groups $E_{7(7)}, E_{7(-5)}$ and $E_{7(-25)}$ by

$$\begin{aligned}E_{7(7)} &= \{\alpha \in \text{Iso}_{\mathbf{R}}(\mathfrak{P}') \mid \alpha(P \times Q)\alpha^{-1} = \alpha P \times \alpha Q\}, \\ E_{7(-5)} &= \{\alpha \in \text{Iso}_C(\mathfrak{P}^C) \mid \alpha(P \times Q)\alpha^{-1} = \alpha P \times \alpha Q, \langle \alpha P, \alpha Q \rangle_\sigma = \langle P, Q \rangle_\sigma\}, \\ E_{7(-25)} &= \{\alpha \in \text{Iso}_{\mathbf{R}}(\mathfrak{P}) \mid \alpha(P \times Q)\alpha^{-1} = \alpha P \times \alpha Q\}.\end{aligned}$$

These groups can also be defined by

$$E_{7(7)} \cong (E_7^C)^{\tau\gamma}, \quad E_{7(-5)} \cong (E_7^C)^{\tau\lambda\sigma}, \quad E_{7(-25)} \cong (E_7^C)^\tau.$$

Theorem 4.15.1. *The polar decompositions of the groups $E_{7(7)}, E_{7(-5)}$ and $E_{7(-25)}$ are respectively given by*

$$\begin{aligned}E_{7(7)} &\simeq SU(8)/\mathbf{Z}_2 \times \mathbf{R}^{70}, \\ E_{7(-5)} &\simeq (SU(2) \times Spin(12))/\mathbf{Z}_2 \times \mathbf{R}^{64}, \\ E_{7(-25)} &\simeq (U(1) \times E_6)/\mathbf{Z}_3 \times \mathbf{R}^{54}.\end{aligned}$$

Proof. These are the facts corresponding to Theorems 4.12.5, 4.11.15 and 4.10.2.

Theorem 4.15.2. *The centers of the groups $E_{7(7)}, E_{7(-5)}$ and $E_{7(-25)}$ are the group of order 2:*

$$z(E_{7(7)}) = \mathbf{Z}_2, \quad z(E_{7(-5)}) = \mathbf{Z}_2, \quad z(E_{7(-25)}) = \mathbf{Z}_2.$$

5. Exceptional Lie group E_8

5.1. Lie algebra \mathfrak{e}_8^C

Theorem 5.1.1. *In a $133 + 56 \times 2 + 3 = 248$ dimensional C -vector space*

$$\mathfrak{e}_8^C = \mathfrak{e}_7^C \oplus \mathfrak{P}^C \oplus \mathfrak{P}^C \oplus C \oplus C \oplus C,$$

if we define a Lie bracket $[R_1, R_2]$ by

$$[(\Phi_1, P_1, Q_1, r_1, s_1, t_1), (\Phi_2, P_2, Q_2, r_2, s_2, t_2)] = (\Phi, P, Q, r, s, t),$$

where

$$\begin{cases} \Phi = [\Phi_1, \Phi_2] + P_1 \times Q_2 - P_2 \times Q_1 \\ P = \Phi_1 P_2 - \Phi_2 P_1 + r_1 P_2 - r_2 P_1 + s_1 Q_2 - s_2 Q_1 \\ Q = \Phi_1 Q_2 - \Phi_2 Q_1 - r_1 Q_2 + r_2 Q_1 + t_1 P_2 - t_2 P_1 \\ r = -\frac{1}{8}\{P_1, Q_2\} + \frac{1}{8}\{P_2, Q_1\} + s_1 t_2 - s_2 t_1 \\ s = \frac{1}{4}\{P_1, P_2\} + 2r_1 s_2 - 2r_2 s_1 \\ t = -\frac{1}{4}\{Q_1, Q_2\} - 2r_1 t_2 + 2r_2 t_1, \end{cases}$$

then \mathfrak{e}_8^C is a C -Lie algebra.

Proof. Among the definition of the Lie algebra, the relations

$$\begin{aligned} [R_1, R_2 + R_3] &= [R_1, R_2] + [R_1, R_3], \\ [kR_1, R_2] &= k[R_1, R_2], \quad k \in C, \\ [R_1, R_2] &= -[R_2, R_1] \end{aligned}$$

are evident, and we are left to show the Jacobi identity, which can be proved by direct calculations as follows.

$$\begin{aligned} &[R_1, [R_2, R_3]] + [R_2, [R_3, R_1]] + [R_3, [R_1, R_2]] \\ &= \dots (\text{using } [\Phi, P \times Q] = \Phi P \times Q + P \times \Phi Q \text{ (Proposition 4.3.2)}, \\ &\quad (P \times R)Q - (Q \times R)P + \frac{1}{8}\{Q, R\}P - \frac{1}{8}\{P, R\}Q - \frac{1}{4}\{P, Q\}R = 0 \\ &\quad (\text{Lemma 4.1.1.(3), } \{\Phi P, Q\} + \{P, \Phi Q\} = 0 \text{ (Proposition 4.2.2.(2)) etc.}) \dots \\ &= 0. \end{aligned}$$

5.2. Simplicity of \mathfrak{e}_8^C

We use the following notation in \mathfrak{e}_8^C :

$$\begin{aligned} \Phi &= (\Phi, 0, 0, 0, 0, 0), & P^- &= (0, P, 0, 0, 0, 0), \\ Q_- &= (0, 0, Q, 0, 0, 0), & r &= (0, 0, 0, r, 0, 0), \\ s^- &= (0, 0, 0, 0, s, 0), & t_- &= (0, 0, 0, 0, 0, t). \end{aligned}$$

Theorem 5.2.1. *The C -Lie algebra \mathfrak{e}_8^C is simple.*

Proof. We use the decomposition of \mathfrak{e}_8^C :

$$\mathfrak{e}_8^C = \mathfrak{e}_7^C \oplus \mathfrak{K}^C,$$

where $\mathfrak{K}^C = \mathfrak{P}^C \oplus \mathfrak{P}^C \oplus C \oplus C \oplus C$. Let $p : \mathfrak{e}_8^C \rightarrow \mathfrak{e}_7^C$ and $q : \mathfrak{e}_8^C \rightarrow \mathfrak{K}^C$ be projections of $\mathfrak{e}_8^C = \mathfrak{e}_7^C \oplus \mathfrak{K}^C$. Now, let \mathfrak{a} be a non-zero ideal of \mathfrak{e}_8^C . Then $p(\mathfrak{a})$ is an ideal of \mathfrak{e}_7^C . Indeed, if $\Phi \in p(\mathfrak{a})$, then there exists $(0, P, Q, r, s, t) \in \mathfrak{K}^C$ such that $(\Phi, P, Q, r, s, t) \in \mathfrak{a}$. For any $\Phi_1 \in \mathfrak{e}_7^C$, we have

$$\mathfrak{a} \ni [\Phi_1, (\Phi, P, Q, r, s, t)] = ([\Phi_1, \Phi], \Phi_1 P, \Phi_1 Q, 0, 0, 0),$$

hence $[\Phi_1, \Phi] \in p(\mathfrak{a})$.

We shall show that either $\mathfrak{e}_7^C \cap \mathfrak{a} \neq \{0\}$ or $\mathfrak{K}^C \cap \mathfrak{a} \neq \{0\}$. Assume that $\mathfrak{e}_7^C \cap \mathfrak{a} = \{0\}$ and $\mathfrak{K}^C \cap \mathfrak{a} = \{0\}$. Then the mapping $p|_{\mathfrak{a}} : \mathfrak{a} \rightarrow \mathfrak{e}_7^C$ is injective because $\mathfrak{K}^C \cap \mathfrak{a} = \{0\}$. Since $p(\mathfrak{a})$ is a non-zero ideal of \mathfrak{e}_7^C and \mathfrak{e}_7^C is simple, we have $p(\mathfrak{a}) = \mathfrak{e}_7^C$. Hence $\dim_C(\mathfrak{a}) = \dim_C(p(\mathfrak{a})) = \dim_C(\mathfrak{e}_7^C) = 133$. On the other hand, since $\mathfrak{e}_7^C \cap \mathfrak{a} = \{0\}$, $q|_{\mathfrak{a}} : \mathfrak{a} \rightarrow \mathfrak{K}^C$ is also injective. Hence we have $\dim_C(\mathfrak{a}) \leq \dim_C(\mathfrak{K}^C) = 56 \times 2 + 3 = 115$. This leads to a contradiction.

We now consider the following two cases.

(1) Case $\mathfrak{e}_7^C \cap \mathfrak{a} \neq \{0\}$. From the simplicity of \mathfrak{e}_7^C , we have $\mathfrak{e}_7^C \cap \mathfrak{a} = \mathfrak{e}_7^C$, hence $\mathfrak{a} \supset \mathfrak{e}_7^C$. On the other hand, we have

$$\begin{aligned} \mathfrak{a} \ni [\Phi(0, 0, 0, 1), (0, 0, 1, 0)^-] &= (0, 0, 1, 0)^-, \\ \mathfrak{a} \ni [\Phi(0, 0, 0, 1), (0, 0, 0, -1)_-] &= (0, 0, 0, 1)_-, \\ \mathfrak{a} \ni [(0, 0, 1, 0)^-, (0, 0, 0, 4)^-] &= 1^-, \\ \mathfrak{a} \ni [(0, 0, 0, 1)_-, (0, 0, 4, 0)_-] &= 1_-, \\ \mathfrak{a} \ni [1^-, 1_-] &= 1, \\ \mathfrak{a} \ni [1^- + 1_-, Q^- + P_-] &= P^- + Q_-, \end{aligned}$$

Therefore, $\mathfrak{a} \supset \mathfrak{e}_7^C \oplus \mathfrak{K}^C = \mathfrak{e}_8^C$ which implies $\mathfrak{a} = \mathfrak{e}_8^C$.

(2) Case $\mathfrak{K}^C \cap \mathfrak{a} \neq \{0\}$. Let $R = (0, P, Q, r, s, t)$ be a non-zero element of $\mathfrak{K}^C \cap \mathfrak{a} \subset \mathfrak{a}$.

(i) Case $R = (0, P, Q, r, s, t)$, $P \neq 0$. We have

$$\begin{aligned} \mathfrak{a} \ni [1, [1_-, [1, R]]] &= [1, [1_-, (0, P, -Q, 0, 2s, -2t)]] \\ &= [1, (0, 0, P, -2s, 0, 0)] = -(0, 0, P, 0, 0, 0) = -P_-. \end{aligned}$$

We choose $P_1 \in \mathfrak{P}^C$ so that $P \times P_1 \neq 0$ (Lemma 4.5.3) and choose $\Phi \in \mathfrak{e}_7^C$ so that $[\Phi, P \times P_1] \neq 0$. (Since \mathfrak{e}_7^C simple, the center of \mathfrak{e}_7^C consists only of 0, so such Φ exists). Then we have

$$\mathfrak{a} \ni [\Phi, [P_1^-, P_-]] = [\Phi, P \times P_1].$$

Hence this case is reduced to the case (1).

(ii) Case $R = (0, P, Q, r, s, t), Q \neq 0$. The argument is similar to (i).

(iii) Case $R = (0, 0, 0, r, s, t), r \neq 0$. For $0 \neq P \in \mathfrak{P}^C$, we have

$$\begin{aligned}\mathfrak{a} \ni [P^-, [1^-, [1_-, R]]] &= [P^-, [1^-, (0, 0, 0, -s, 0, 2r)]] \\ &= [P^-, (0, 0, 0, 2r, 2s, 0)] = (0, -2rP, 0, 0, 0, 0).\end{aligned}$$

Hence this case is reduced to the case (ii) above.

(iv) Case $R = (0, 0, 0, 0, s, t), s \neq 0$. We have

$$\mathfrak{a} \ni [1_-, R] = (0, 0, 0, -s, 0, 0).$$

Hence this case is reduced to the case (iii) above.

(v) Case $R = (0, 0, 0, 0, 0, t), t \neq 0$. The argument is similar to (iv).

Consequently, we have $\mathfrak{a} = \mathfrak{e}_8^C$, which proves the simplicity of \mathfrak{e}_8^C .

For $R \in \mathfrak{e}_8^C$ we denote $\text{ad}R : \mathfrak{e}_8^C \rightarrow \mathfrak{e}_8^C$, that is, $\text{ad}R(R_1) = [R, R_1]$, by $\Theta(R) = \text{ad}R$. Since \mathfrak{e}_8^C is simple (Theorem 5.2.1), we obtain an isomorphism of Lie algebras

$$\mathfrak{e}_8^C \cong \Theta(\mathfrak{e}_8^C) = \{\Theta(R) \mid R \in \mathfrak{e}_8^C\}$$

by assigning $\Theta(R)$ to R . Moreover, \mathfrak{e}_8^C is isomorphic to the algebra

$$\text{Der}(\mathfrak{e}_8^C) = \{\Theta \in \text{Hom}_C(\mathfrak{e}_8^C) \mid \Theta[R_1, R_2] = [\Theta R_1, R_2] + [R_2, \Theta R_2]\}.$$

Hereafter we often denote $\Theta(R)$ by R identifying $\mathfrak{e}_8^C \cong \text{Der}(\mathfrak{e}_8^C)$.

5.3. Killing form of \mathfrak{e}_8^C

Definition. We define a symmetric inner product $(R_1, R_2)_8$ in \mathfrak{e}_8^C by

$$(R_1, R_2)_8 = (\Phi_1, \Phi_2)_7 - \{Q_1, P_2\} + \{P_1, Q_2\} - 8r_1r_2 - 4t_1s_2 - s_1t_2,$$

where $R_i = (\Phi_i, P_i, Q_i, r_i, s_i, t_i) \in \mathfrak{e}_8^C$.

Lemma 5.3.1. *The inner product $(R_1, R_2)_8$ of \mathfrak{e}_8^C is \mathfrak{e}_8^C -adjoint invariant:*

$$([R, R_1], R_2)_8 + (R_1, [R, R_2])_8 = 0, \quad R, R_i \in \mathfrak{e}_8^C.$$

Proof. $([R, R_1], R_2)_8$

$$\begin{aligned}
&= \left(\begin{array}{c} [\Phi, \Phi_1] + P \times Q_1 - P_1 \times Q \\ \Phi P_1 - \Phi_1 P + r P_1 - r_1 P + s Q_1 - s_1 Q \\ \Phi Q_1 - \Phi_1 Q - r Q_1 + r_1 Q + t P_1 - t_1 P \\ -\frac{1}{8}\{P, Q_1\} + \frac{1}{8}\{P_1, Q\} + s t_1 - s_1 t \\ \frac{1}{4}\{P, P_1\} + 2r s_1 - 2r_1 s \\ -\frac{1}{4}\{Q, Q_1\} - 2r t_1 + 2r_1 t \end{array} \right), \left(\begin{array}{c} \Phi_2 \\ P_2 \\ Q_2 \\ r_2 \\ s_2 \\ t_2 \end{array} \right)_8 \\
&= \cdots (\text{using } ([\Phi, \Phi_1], \Phi_2)_7 + (\Phi_1, [\Phi, \Phi_2])_7 = 0 \text{ (Lemma 4.5.1.(1))}, \\
&\quad (\Phi, P \times Q)_7 = \{\Phi P, Q\} \text{ (Lemma 4.5.1.(2)) etc.}) \cdots \\
&= -(R_1, [R, R_2])_8.
\end{aligned}$$

Theorem 5.3.2. *The Killing form B_8 of the Lie algebra \mathfrak{e}_8^C is given by*

$$\begin{aligned}
&B_8(R_1, R_2) \\
&= -15(R_1, R_2)_8 \\
&= -15(\Phi_1, \Phi_2)_7 + 15\{Q_1, P_2\} - 15\{P_1, Q_2\} + 120r_1r_2 + 60t_1s_2 + 60s_1t_2 \\
&= \frac{5}{3}B_7(\Phi_1, \Phi_2) + 15\{Q_1, P_2\} - 15\{P_1, Q_2\} + 120r_1r_2 + 60t_1s_2 + 60s_1t_2,
\end{aligned}$$

where $R_i = (\Phi_i, P_i, Q_i, r_i, s_i, t_i) \in \mathfrak{e}_8^C$ and B_7 is the Killing form of \mathfrak{e}_7^C .

Proof. Since \mathfrak{e}_8^C is simple (Theorem 5.2.1), there exist $k \in C$ such that

$$B_8(R_1, R_2) = k(R_1, R_2)_8, \quad R_i \in \mathfrak{e}_8^C.$$

To determine k , let $R_1 = R_2 = (0, 0, 0, 1, 0, 0) = 1$. Then, we have

$$(1, 1)_8 = -8.$$

On the other hand, since

$$[1, [1, (\Phi, P, Q, r, s, t)]] = [1, (0, P, -Q, 0, 2s, -2t)] = (0, P, Q, 0, 4s, 4t),$$

we have

$$B_8(1, 1) = 56 \times 2 + 4 \times 2 = 120.$$

Therefore $k = -15$. Thus we have $B_8(R_1, R_2) = -15(R_1, R_2)_8$.

5.4. Complex exceptional Lie group E_8^C

Definition. The group E_8^C is defined to be the automorphism group of the Lie algebra \mathfrak{e}_8^C :

$$E_8^C = \{\alpha \in \text{Iso}_C(\mathfrak{e}_8^C) \mid \alpha[R_1, R_2] = [\alpha R_1, \alpha R_2]\}.$$

Theorem 5.4.1. *The group E_8^C is connected.*

Proof. Denote by $\text{Inn}(\mathfrak{e}_8^C)$ the subgroup generated by the inner automorphisms $\exp(\Theta(R))$, $R \in \mathfrak{e}_8^C$ in the automorphism group $\text{Aut}(\mathfrak{e}_8^C) = E_8^C$ of the Lie algebra \mathfrak{e}_8^C . It is known that $\text{Aut}(\mathfrak{e}_8^C)/\text{Inn}(\mathfrak{e}_8^C) = \{1\}$ holds for the C -algebra of E_8 type (see for example, Matsushima [19]), that is,

$$\text{Aut}(\mathfrak{e}_8^C) = \text{Inn}(\mathfrak{e}_8^C).$$

Since $\text{Inn}(\mathfrak{e}_8^C)$ is connected, E_8^C which is equal to $\text{Inn}(\mathfrak{e}_8^C)$ is also connected.

Remark. We can also prove the connectedness of the group E_8^C from the following fact. For $R \in \mathfrak{e}_8^C$, we define a C -linear mapping $R \times R : \mathfrak{e}_8^C \rightarrow \mathfrak{e}_8^C$ by

$$(R \times R)R_1 = \Theta(R)^2 R_1 + \frac{1}{30} B_8(R, R_1)R, \quad R_1 \in \mathfrak{e}_8^C,$$

and we define a space \mathfrak{W}^C by

$$\mathfrak{W}^C = \{R \in \mathfrak{e}_8^C \mid R \times R = 0, R \neq 0\}.$$

Then we have

$$E_8^C / (E_8^C)_{1-} \simeq \mathfrak{W}^C,$$

where $(E_8^C)_{1-} = \{\alpha \in E_8^C \mid \alpha 1_- = 1_-\} = \exp(\Phi(0, 0, \mathfrak{P}^C, 0, 0, C))E_7$. The connectedness of E_8^C follows from the connectedness of $(E_8^C)_{1-}$ and \mathfrak{W}^C . (See Imai and Yokota [13]).

The Lie algebra of the group E_8^C is $\text{Der}(\mathfrak{e}_8^C) \cong \mathfrak{e}_8^C$, and therefore we have shown that E_8^C is a complex Lie group of type E_8 , since we will show in Theorem 5.6.2 that \mathfrak{e}_8^C is a Lie algebra of type E_8 . It is known by the general theory of Lie groups, that if a complex Lie group of type E_8 is connected, then it is simply connected, and hence we have obtained the following result.

Theorem 5.4.2. *E_8^C is a simply connected complex Lie group of type E_8 .*

5.5. Compact exceptional Lie group E_8

We define C -linear transformations λ, λ' of \mathfrak{e}_8^C respectively by

$$\begin{aligned} \lambda(\Phi, P, Q, r, s, t) &= (\lambda\Phi\lambda^{-1}, \lambda P, \lambda Q, r, s, t), \\ \lambda'(\Phi, P, Q, r, s, t) &= (\Phi, Q, -P, -r, -t, -s), \end{aligned}$$

where λ in the right hand side is the same as $\lambda \in E_7$ defined in Section 4.3. The mappings λ and λ' preserve the Lie bracket in \mathfrak{e}_8^C , that is, $\lambda, \lambda' \in \text{Aut}(\mathfrak{e}_8^C) = E_8^C$. We set

$$\tilde{\lambda} = \lambda\lambda' = \lambda'\lambda.$$

Finally, we denote by τ the complex conjugation in \mathfrak{e}_8^C , that is,

$$\tau(\Phi, P, Q, r, s, t) = (\tau\Phi, \tau P, \tau Q, \tau r, \tau s, \tau t),$$

where τ in the right hand side is the usual complex conjugation in the complexification.

Definition. We define a Hermitian inner product $\langle R_1, R_2 \rangle$ in \mathfrak{e}_8^C by

$$\langle R_1, R_2 \rangle = -\frac{1}{15}B_8(\tau\tilde{\lambda}R_1, R_2).$$

Proposition 5.5.1. *The Hermitian inner product $\langle R_1, R_2 \rangle$ in \mathfrak{e}_8^C is positive definite.*

Proof. Let $R_i = (\Phi_i, P_i, Q_i, r_i, s_i, t_i) \in \mathfrak{e}_8^C, i = 1, 2$. Since $\tau\tilde{\lambda}R_1 = (\tau\lambda\Phi_1\lambda^{-1}\tau, \tau\lambda Q_1, -\tau\lambda P_1, -\tau r_1, -\tau t_1, -\tau s_1)$, we have, by Theorem 5.3.2, that

$$\begin{aligned} \langle R_1, R_2 \rangle &= (\tau\lambda\Phi_1\lambda^{-1}\tau, \Phi_2)_7 + \langle P_1, P_2 \rangle + \langle Q_1, Q_2 \rangle + 8(\tau r_1)r_2 + 4(\tau s_1)s_2 + 4(\tau t_1)t_2. \end{aligned}$$

Hence, it is sufficient to show that $(\tau\lambda\Phi_1\lambda^{-1}\tau, \Phi_2)_7$ is positive definite. Let $\Phi_i = \Phi(\phi_i, A_i, B_i, \nu_i), i = 1, 2$. Since $\tau\lambda\Phi_1\lambda^{-1}\tau = \Phi(-\tau^t\phi_1\tau, -\tau B_1, -\tau A_1, -\tau\nu_1)$, we have

$$(\tau\lambda\Phi_1\lambda^{-1}\tau, \Phi_2)_7 = 2(\tau^t\phi_1\tau, \phi_2)_6 + 4\langle A_1, A_2 \rangle + 4\langle B_1, B_2 \rangle + \frac{8}{3}(\tau\nu_1)\nu_2.$$

Therefore, it is enough to show that $(\tau^t\phi_1\tau, \phi_2)_6$ is positive definite. Let $\phi_i = \delta_i + \tilde{T}_i \in \mathfrak{e}_6^C, \delta_i \in \mathfrak{f}_4^C, \tilde{T}_i \in \mathfrak{J}_0^C, i = 1, 2$. Since $\tau^t\phi_1\tau = -\tau\delta_1\tau + \tau\tilde{T}_1$, we have

$$(\tau^t\phi_1\tau, \phi_2)_6 = -(\tau\delta_1\tau, \delta_2)_4 + \langle T_1, T_2 \rangle.$$

Consequently, it is sufficient to show that $-(\tau\delta_1\tau, \delta_2)_4$ is positive definite, which can be seen, however, from the fact that the following set

$$\sqrt{2}[\tilde{E}_1, \tilde{F}_2(e_i)], \quad \sqrt{2}[\tilde{E}_1, \tilde{F}_3(e_i)], \quad \sqrt{2}[\tilde{E}_3, \tilde{F}_1(e_i)], \quad 0 \leq i \leq 7,$$

$$\frac{1}{\sqrt{2}}[\tilde{F}_1(e_i), \tilde{F}_1(e_j)], \quad 0 \leq i < j \leq 7$$

forms an orthonormal C -basis of \mathfrak{f}_4^C with respect to the inner product $-(\tau\delta_1\tau, \delta_2)_4$. Thus the proposition is proved.

Definition. We define a group E_8 by

$$\begin{aligned} E_8 &= \{\alpha \in E_8^C \mid \langle \alpha R_1, \alpha R_2 \rangle = \langle R_1, R_2 \rangle\} \\ &= \{\alpha \in E_8^C \mid \tau\tilde{\lambda}\alpha = \alpha\tau\tilde{\lambda}\}. \end{aligned}$$

Theorem 5.5.2. *The group E_8 is a compact Lie group.*

Proof. E_8 is a compact Lie group as a closed subgroup of the unitary group

$$U(248) = U(\mathfrak{e}_8^C) = \{\alpha \in \text{Iso}_C(\mathfrak{e}_8^C) \mid \langle \alpha R_1, \alpha R_2 \rangle = \langle R_1, R_2 \rangle\}.$$

Theorem 5.5.3. *The Lie algebra \mathfrak{e}_8 of the group E_8 is*

$$\begin{aligned} \mathfrak{e}_8 &= \{R \in \mathfrak{e}_8^C \mid \tau \tilde{\lambda} R = R\} \\ &= \{(\Phi, P, -\tau \lambda P, r, s, -\tau s) \in \mathfrak{e}_8^C \mid \Phi \in \mathfrak{e}_7, P \in \mathfrak{P}^C, r \in i\mathbf{R}, s \in C\}. \end{aligned}$$

Proof. For $R = (\Phi, P, Q, r, s, t) \in \mathfrak{e}_8^C$, since

$$\tau \tilde{\lambda} R = (\tau \lambda \Phi \lambda^{-1} \tau, \tau \lambda Q, -\tau \lambda P, -\tau r, -\tau t, -\tau s).$$

the condition $\tau \tilde{\lambda} R = R$ is equivalent to $\tau \lambda \Phi = \Phi \lambda \tau, Q = -\tau \lambda P, \tau r = -r, t = -\tau s$, hence we have the theorem.

Proposition 5.5.4. *The complexification of the Lie algebra \mathfrak{e}_8 is \mathfrak{e}_8^C . Hence \mathfrak{e}_8 is simple.*

Proof. For $R \in \mathfrak{e}_8^C$, the conjugate transposed mapping R^* of R with respect to the inner product $\langle R_1, R_2 \rangle$ of \mathfrak{e}_8^C is $R^* = \tau \tilde{\lambda} R \tilde{\lambda} \tau \in \mathfrak{e}_8^C$, and for $R \in \mathfrak{e}_8^C$, R belongs to \mathfrak{e}_8 if and only if $R^* = -R$. Now, any element $R \in \mathfrak{e}_8^C$ is represented by

$$R = \frac{R - R^*}{2} + i \frac{R + R^*}{2i}, \quad \frac{R - R^*}{2}, \frac{R + R^*}{2i} \in \mathfrak{e}_8.$$

Hence \mathfrak{e}_8^C is the complexification of \mathfrak{e}_8 . Since \mathfrak{e}_8^C is simple (Theorem 5.2.1), \mathfrak{e}_8 is also simple.

Analogously as in \mathfrak{e}_8^C , for $R \in \mathfrak{e}_8$, we identify R with $\Theta(R)$ and regard $\mathfrak{e}_8 \cong \Theta(\mathfrak{e}_8)$.

Theorem 5.5.5. *The polar decomposition of the Lie group E_8^C is given by*

$$E_8^C \simeq E_8 \times \mathbf{R}^{248}.$$

In particular, the group E_8 is simply connected.

Proof. Evidently E_8^C is an algebraic subgroup of $\text{Iso}_C(\mathfrak{e}_8^C) = GL(248, C)$. For $\alpha \in E_8^C$, the conjugate transposed mapping α^* of α with respect to the inner product $\langle R_1, R_2 \rangle$ is $\alpha^* = \tau \tilde{\lambda} \alpha^{-1} \tilde{\lambda} \tau \in E_8^C$. Therefore, by Chevalley's lemma, we have

$$E_8^C \simeq (E_8^C \cap U(\mathfrak{e}_8^C)) \times \mathbf{R}^d = E_8 \times \mathbf{R}^d, \quad d = 248.$$

Since E_8^C is simply connected (Theorem 5.4.2), E_8 is also simply connected.

5.6 Roots of \mathfrak{e}_8^C

Theorem 5.6.1. *The rank of the Lie algebra \mathfrak{e}_8^C is 8. The roots of \mathfrak{e}_8^C relative to some Cartan subalgebra are given by*

$$\begin{aligned}
& \pm(\lambda_k - \lambda_l), \quad \pm(\lambda_k + \lambda_l), \quad 0 \leq k < l \leq 3, \\
& \pm\lambda_k \pm \frac{1}{2}(\mu_2 - \mu_3), \quad 0 \leq k \leq 3, \\
& \pm\frac{1}{2}(-\lambda_0 - \lambda_1 + \lambda_2 - \lambda_3) \pm \frac{1}{2}(\mu_3 - \mu_1), \\
& \pm\frac{1}{2}(-\lambda_0 + \lambda_1 + \lambda_2 - \lambda_3) \pm \frac{1}{2}(\mu_3 - \mu_1), \\
& \pm\frac{1}{2}(-\lambda_0 + \lambda_1 + \lambda_2 + \lambda_3) \pm \frac{1}{2}(\mu_3 - \mu_1), \\
& \pm\frac{1}{2}(\lambda_0 - \lambda_1 + \lambda_2 + \lambda_3) \pm \frac{1}{2}(\mu_3 - \mu_1), \\
& \pm\frac{1}{2}(\lambda_0 - \lambda_1 + \lambda_2 - \lambda_3) \pm \frac{1}{2}(\mu_1 - \mu_2), \\
& \pm\frac{1}{2}(-\lambda_0 + \lambda_1 + \lambda_2 - \lambda_3) \pm \frac{1}{2}(\mu_1 - \mu_2), \\
& \pm\frac{1}{2}(\lambda_0 + \lambda_1 + \lambda_2 + \lambda_3) \pm \frac{1}{2}(\mu_1 - \mu_2), \\
& \pm\frac{1}{2}(-\lambda_0 - \lambda_1 + \lambda_2 + \lambda_3) \pm \frac{1}{2}(\mu_1 - \mu_2), \\
& \pm\left(\mu_k + \frac{2}{3}\nu\right), \quad 0 \leq k < l \leq 3, \\
& \pm\lambda_k \pm \left(\frac{1}{2}\mu_1 - \frac{2}{3}\nu\right), \quad 0 \leq k \leq 3, \\
& \pm\frac{1}{2}(-\lambda_0 - \lambda_1 + \lambda_2 - \lambda_3) \pm \left(\frac{1}{2}\mu_2 - \frac{2}{3}\nu\right), \\
& \pm\frac{1}{2}(\lambda_0 + \lambda_1 + \lambda_2 - \lambda_3) \pm \left(\frac{1}{2}\mu_2 - \frac{2}{3}\nu\right), \\
& \pm\frac{1}{2}(-\lambda_0 + \lambda_1 + \lambda_2 + \lambda_3) \pm \left(\frac{1}{2}\mu_2 - \frac{2}{3}\nu\right), \\
& \pm\frac{1}{2}(\lambda_0 - \lambda_1 + \lambda_2 + \lambda_3) \pm \left(\frac{1}{2}\mu_2 - \frac{2}{3}\nu\right), \\
& \pm\frac{1}{2}(\lambda_0 - \lambda_1 + \lambda_2 - \lambda_3) \pm \left(\frac{1}{2}\mu_3 - \frac{2}{3}\nu\right), \\
& \pm\frac{1}{2}(-\lambda_0 + \lambda_1 + \lambda_2 - \lambda_3) \pm \left(\frac{1}{2}\mu_3 - \frac{2}{3}\nu\right), \\
& \pm\frac{1}{2}(\lambda_0 + \lambda_1 + \lambda_2 + \lambda_3) \pm \left(\frac{1}{2}\mu_3 - \frac{2}{3}\nu\right), \\
& \pm\frac{1}{2}(-\lambda_0 - \lambda_1 + \lambda_2 + \lambda_3) \pm \left(\frac{1}{2}\mu_3 - \frac{2}{3}\nu\right), \\
& \pm\left(\mu_j - \frac{1}{3}\nu + r\right), \quad 1 \leq j \leq 3, \\
& \pm\lambda_k \pm \left(\frac{1}{2}\mu_1 + \frac{1}{3}\nu\right) \pm r, \quad 0 \leq k \leq 3, \\
& \pm\frac{1}{2}(\lambda_0 + \lambda_1 - \lambda_2 - \lambda_3) \pm \left(\frac{1}{2}\mu_2 + \frac{1}{3}\nu\right) \pm r, \\
& \pm\frac{1}{2}(\lambda_0 + \lambda_1 - \lambda_2 - \lambda_3) \pm \left(\frac{1}{2}\mu_2 + \frac{1}{3}\nu\right) \pm r,
\end{aligned}$$

$$\begin{aligned}
& \pm \frac{1}{2}(\lambda_0 + \lambda_1 + \lambda_2 - \lambda_3) \pm \left(\frac{1}{2}\mu_2 + \frac{1}{3}\nu\right) \pm r, \\
& \pm \frac{1}{2}(\lambda_0 + \lambda_1 - \lambda_2 + \lambda_3) \pm \left(\frac{1}{2}\mu_2 + \frac{1}{3}\nu\right) \pm r, \\
& \pm \frac{1}{2}(\lambda_0 + \lambda_1 + \lambda_2 + \lambda_3) \pm \left(\frac{1}{2}\mu_3 + \frac{1}{3}\nu\right) \pm r, \\
& \pm \frac{1}{2}(\lambda_0 + \lambda_1 - \lambda_2 - \lambda_3) \pm \left(\frac{1}{2}\mu_3 + \frac{1}{3}\nu\right) \pm r, \\
& \pm \frac{1}{2}(\lambda_0 - \lambda_1 + \lambda_2 + \lambda_3) \pm \left(\frac{1}{2}\mu_3 + \frac{1}{3}\nu\right) \pm r, \\
& \pm \frac{1}{2}(\lambda_0 - \lambda_1 - \lambda_2 + \lambda_3) \pm \left(\frac{1}{2}\mu_3 + \frac{2}{3}\nu\right) \pm r, \\
& \pm 2r, \\
& \pm \nu \pm r
\end{aligned}$$

with $\mu_1 + \mu_2 + \mu_3 = 0$.

Proof. Let \mathfrak{h}_7 be the Cartan subalgebra of \mathfrak{e}_7^C given in Theorem 4.6.1, then

$$\mathfrak{h} = \{(\Phi(h), 0, 0, r, 0, 0) \mid \Phi(h) \in \mathfrak{h}_7, r \in C\}$$

is an abelian subalgebra of \mathfrak{e}_8^C (it will be a Cartan subalgebra of \mathfrak{e}_8^C).

I The roots of \mathfrak{e}_7^C are also roots of \mathfrak{e}_8^C . Indeed, we have

$$[(\Phi(h), 0, 0, r, 0, 0), (\Phi, 0, 0, 0, 0, 0)] = ([\Phi(h), \Phi], 0, 0, 0, 0, 0).$$

II We have

$$[(\Phi(h), 0, 0, r, 0, 0), (0, P, 0, 0, 0, 0)] = (0, (\Phi(h) + r)P, 0, 0, 0, 0),$$

and using the same notation as in Theorem 4.6.1, we also have

$$\begin{aligned}
& (\Phi(h_\delta + \tilde{H}, 0, 0, \nu) + r1)(X, Y, \xi, \eta) \\
& = \left(\left(h_\delta + \tilde{H} - \frac{1}{3}\nu + r \right) X, \left(h_\delta - \tilde{H} + \frac{1}{3}\nu + r \right) Y, (\nu + r)\xi, (-\nu + r)\eta \right).
\end{aligned}$$

By putting $Y = 0, \xi = \eta = 0$, we obtain

$$\begin{aligned}
& \text{the root } \mu_k - \frac{1}{3}\nu + r \quad \text{by letting } X = E_k, \\
& \text{the root } \pm \lambda_k - \frac{1}{2}\mu_1 - \frac{1}{3}\nu + r \quad \text{by letting } X = F_1(a), a = e_k \pm ie_{4+k}.
\end{aligned}$$

We can also obtain roots by letting $X = F_2(a), F_3(a)$.

By putting $X = 0, \xi = \eta = 0$ and further $Y = E_k, Y = F_i(a)$, we can again roots.

By putting $X = Y = 0, \xi = 1, \eta = 0$, we can again the root $\nu + r$. Similarly, we can obtain the root $-\nu + r$.

By using

$$[(\Phi(h), 0, 0, r, 0, 0), (0, 0, Q, 0, 0, 0)] = (0, 0, (\Phi(h) - r)Q, 0, 0, 0),$$

we can also obtain roots.

III From the relations

$$\begin{aligned} [(\Phi(h), 0, 0, r, 0), (0, 0, 0, 0, 1, 0)] &= (0, 0, 0, 0, 2r, 0), \\ [(\Phi(h), 0, 0, r, 0), (0, 0, 0, 0, 0, 1)] &= (0, 0, 0, 0, 0, -2r), \end{aligned}$$

we obtain the roots $2r$ and $-2r$.

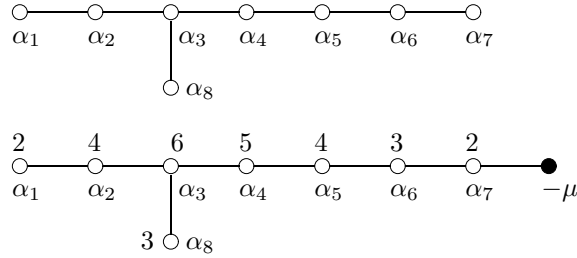
Theorem 5.6.2. *In the root system of Theorem 5.6.1,*

$$\begin{aligned} \alpha_1 &= \frac{1}{2}(\lambda_0 - \lambda_1 - \lambda_2 - \lambda_3) + \frac{1}{2}(\mu_3 - \mu_1), \\ \alpha_2 &= \mu_1 - \frac{1}{3}\nu - r, \quad \alpha_3 = 2r, \\ \alpha_4 &= \mu_2 - \frac{1}{3}\nu - r, \quad \alpha_5 = \lambda_3 - \frac{1}{2}(\mu_2 - \mu_3), \\ \alpha_6 &= \lambda_2 - \lambda_3, \quad \alpha_7 = \lambda_1 - \lambda_2, \quad \alpha_8 = \nu - r \end{aligned}$$

is a fundamental root system of the Lie algebra \mathfrak{e}_8^C and

$$\mu = 2\alpha_1 + 4\alpha_2 + 6\alpha_3 + 5\alpha_4 + 4\alpha_5 + 3\alpha_6 + 2\alpha_7 + 3\alpha_8$$

is the highest root. The Dynkin diagram and the extended Dynkin diagram of \mathfrak{e}_8^C are respectively given by



Proof. In the following, the notation $n_1n_2 \cdots n_8$ denotes the root $n_1\alpha_1 + n_2\alpha_2 + \cdots + n_8\alpha_8$. Now, all positive roots of \mathfrak{e}_8^C are expressed by

$$\begin{array}{ll} \lambda_0 - \lambda_1 = 2 & 3 \quad 4 \quad 3 \quad 2 \quad 1 \quad 0 \quad 2 & \lambda_0 + \lambda_1 = 2 & 4 \quad 6 \quad 5 \quad 4 \quad 3 \quad 2 \quad 3 \\ \lambda_0 - \lambda_2 = 2 & 3 \quad 4 \quad 3 \quad 2 \quad 1 \quad 1 \quad 2 & \lambda_0 + \lambda_2 = 2 & 4 \quad 6 \quad 5 \quad 4 \quad 3 \quad 1 \quad 3 \\ \lambda_0 - \lambda_3 = 2 & 3 \quad 4 \quad 3 \quad 2 \quad 2 \quad 1 \quad 2 & \lambda_0 + \lambda_3 = 2 & 4 \quad 6 \quad 5 \quad 4 \quad 2 \quad 1 \quad 3 \\ \lambda_1 - \lambda_2 = 0 & 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1 \quad 0 & \lambda_1 + \lambda_2 = 0 & 1 \quad 2 \quad 2 \quad 2 \quad 2 \quad 1 \quad 1 \\ \lambda_1 - \lambda_3 = 0 & 0 \quad 0 \quad 0 \quad 0 \quad 1 \quad 1 \quad 0 & \lambda_1 + \lambda_3 = 0 & 1 \quad 2 \quad 2 \quad 2 \quad 1 \quad 1 \quad 1 \\ \lambda_2 - \lambda_3 = 0 & 0 \quad 0 \quad 0 \quad 0 \quad 1 \quad 0 \quad 0 & \lambda_2 + \lambda_3 = 0 & 1 \quad 2 \quad 2 \quad 2 \quad 1 \quad 0 \quad 1 \end{array}$$

$$\begin{aligned} \lambda_0 + \frac{1}{2}(\mu_2 - \mu_3) &= 2 \quad 4 \quad 6 \quad 5 \quad 3 \quad 2 \quad 1 \quad 3 \\ \lambda_1 + \frac{1}{2}(\mu_2 - \mu_3) &= 0 \quad 1 \quad 2 \quad 2 \quad 1 \quad 1 \quad 1 \quad 1 \end{aligned}$$

$$\begin{aligned}
\lambda_2 + \frac{1}{2}(\mu_2 - \mu_3) &= 0 & 1 & 2 & 2 & 1 & 1 & 0 & 1 \\
\lambda_3 + \frac{1}{2}(\mu_2 - \mu_3) &= 0 & 1 & 2 & 2 & 1 & 0 & 0 & 1 \\
\lambda_0 - \frac{1}{2}(\mu_2 - \mu_3) &= 2 & 3 & 4 & 3 & 3 & 2 & 1 & 2 \\
\lambda_1 - \frac{1}{2}(\mu_2 - \mu_3) &= 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
\lambda_2 - \frac{1}{2}(\mu_2 - \mu_3) &= 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
\lambda_3 - \frac{1}{2}(\mu_2 - \mu_3) &= 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
\frac{1}{2}(\lambda_0 + \lambda_1 + \lambda_2 - \lambda_3) + \frac{1}{2}(\mu_3 - \mu_1) &= 1 & 1 & 2 & 2 & 2 & 2 & 1 & 1 \\
\frac{1}{2}(\lambda_0 + \lambda_1 - \lambda_2 + \lambda_3) + \frac{1}{2}(\mu_3 - \mu_1) &= 1 & 1 & 2 & 2 & 2 & 1 & 1 & 1 \\
\frac{1}{2}(\lambda_0 - \lambda_1 + \lambda_2 + \lambda_3) + \frac{1}{2}(\mu_3 - \mu_1) &= 1 & 1 & 2 & 2 & 2 & 1 & 0 & 1 \\
\frac{1}{2}(\lambda_0 - \lambda_1 - \lambda_2 - \lambda_3) + \frac{1}{2}(\mu_3 - \mu_1) &= 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{2}(\lambda_0 + \lambda_1 + \lambda_2 - \lambda_3) - \frac{1}{2}(\mu_3 - \mu_1) &= 1 & 3 & 4 & 3 & 2 & 2 & 1 & 2 \\
\frac{1}{2}(\lambda_0 + \lambda_1 - \lambda_2 + \lambda_3) - \frac{1}{2}(\mu_3 - \mu_1) &= 1 & 3 & 4 & 3 & 2 & 1 & 1 & 2 \\
\frac{1}{2}(\lambda_0 - \lambda_1 + \lambda_2 + \lambda_3) - \frac{1}{2}(\mu_3 - \mu_1) &= 1 & 3 & 4 & 3 & 2 & 1 & 0 & 2 \\
\frac{1}{2}(\lambda_0 - \lambda_1 - \lambda_2 - \lambda_3) - \frac{1}{2}(\mu_3 - \mu_1) &= 1 & 2 & 2 & 1 & 0 & 0 & 0 & 1 \\
\frac{1}{2}(\lambda_0 - \lambda_1 + \lambda_2 - \lambda_3) + \frac{1}{2}(\mu_1 - \mu_2) &= 1 & 2 & 2 & 1 & 1 & 1 & 0 & 1 \\
\frac{1}{2}(\lambda_0 - \lambda_1 - \lambda_2 + \lambda_3) + \frac{1}{2}(\mu_1 - \mu_2) &= 1 & 2 & 2 & 1 & 1 & 0 & 0 & 1 \\
\frac{1}{2}(\lambda_0 + \lambda_1 - \lambda_2 - \lambda_3) + \frac{1}{2}(\mu_1 - \mu_2) &= 1 & 2 & 2 & 1 & 1 & 1 & 1 & 1 \\
\frac{1}{2}(\lambda_0 + \lambda_1 + \lambda_2 + \lambda_3) + \frac{1}{2}(\mu_1 - \mu_2) &= 1 & 3 & 4 & 3 & 3 & 2 & 1 & 2 \\
\frac{1}{2}(\lambda_0 - \lambda_1 + \lambda_2 - \lambda_3) - \frac{1}{2}(\mu_1 - \mu_2) &= 1 & 1 & 2 & 2 & 1 & 1 & 0 & 1 \\
\frac{1}{2}(\lambda_0 - \lambda_1 - \lambda_2 + \lambda_3) - \frac{1}{2}(\mu_1 - \mu_2) &= 1 & 1 & 2 & 2 & 1 & 0 & 0 & 1 \\
\frac{1}{2}(\lambda_0 + \lambda_1 - \lambda_2 - \lambda_3) - \frac{1}{2}(\mu_1 - \mu_2) &= 1 & 1 & 2 & 2 & 1 & 1 & 1 & 1 \\
\frac{1}{2}(\lambda_0 + \lambda_1 + \lambda_2 + \lambda_3) - \frac{1}{2}(\mu_1 - \mu_2) &= 1 & 2 & 4 & 4 & 3 & 2 & 1 & 2 \\
\mu_1 + \frac{2}{3}\nu &= 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\
\mu_2 + \frac{2}{3}\nu &= 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\
-\mu_3 - \frac{2}{3}\nu &= 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0
\end{aligned}$$

$$\lambda_0 + \frac{1}{2}\mu_1 - \frac{2}{3}\nu = 2 \quad 4 \quad 5 \quad 4 \quad 3 \quad 2 \quad 1 \quad 2$$

$$\lambda_0 - \frac{1}{2}\mu_1 + \frac{2}{3}\nu = 2 \quad 3 \quad 5 \quad 4 \quad 3 \quad 2 \quad 1 \quad 3$$

$$\lambda_1 + \frac{1}{2}\mu_1 - \frac{2}{3}\nu = 0 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 0$$

$$\lambda_1 - \frac{1}{2}\mu_1 + \frac{2}{3}\nu = 0 \quad 0 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1$$

$$\lambda_2 + \frac{1}{2}\mu_1 - \frac{2}{3}\nu = 0 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 0 \quad 0$$

$$\lambda_2 - \frac{1}{2}\mu_1 + \frac{2}{3}\nu = 0 \quad 0 \quad 1 \quad 1 \quad 1 \quad 1 \quad 0 \quad 1$$

$$\lambda_3 + \frac{1}{2}\mu_1 - \frac{2}{3}\nu = 0 \quad 1 \quad 1 \quad 1 \quad 1 \quad 0 \quad 0 \quad 0$$

$$\lambda_3 - \frac{1}{2}\mu_1 + \frac{2}{3}\nu = 0 \quad 0 \quad 1 \quad 1 \quad 1 \quad 0 \quad 0 \quad 1$$

$$\frac{1}{2}(\lambda_0 + \lambda_1 + \lambda_2 - \lambda_3) + \frac{1}{2}\mu_2 - \frac{2}{3}\nu = 1 \quad 2 \quad 3 \quad 3 \quad 2 \quad 2 \quad 1 \quad 1$$

$$\frac{1}{2}(\lambda_0 + \lambda_1 - \lambda_2 + \lambda_3) + \frac{1}{2}\mu_2 - \frac{2}{3}\nu = 1 \quad 2 \quad 3 \quad 3 \quad 2 \quad 1 \quad 1 \quad 1$$

$$\frac{1}{2}(\lambda_0 - \lambda_1 + \lambda_2 + \lambda_3) + \frac{1}{2}\mu_2 - \frac{2}{3}\nu = 1 \quad \begin{array}{cccccc} 2 & 3 & 3 & 2 & 1 & 0 & 1 \end{array}$$

$$\frac{1}{2}(\lambda_0 - \lambda_1 - \lambda_2 - \lambda_3) + \frac{1}{2}\mu_2 - \frac{2}{3}\nu = 1 \quad 1 \quad 1 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0$$

$$\frac{1}{2}(\lambda_0 + \lambda_1 + \lambda_2 - \lambda_3) - \frac{1}{2}\mu_2 + \frac{2}{3}\nu = 1 \quad 2 \quad 3 \quad 2 \quad 2 \quad 2 \quad 1 \quad 2$$

$$\frac{1}{2}(\lambda_0 + \lambda_1 - \lambda_2 + \lambda_3) - \frac{1}{2}\mu_2 + \frac{2}{3}\nu = 1 \quad 2 \quad 3 \quad 2 \quad 2 \quad 1 \quad 1 \quad 2$$

$$\frac{1}{2}(\lambda_0 - \lambda_1 + \lambda_2 + \lambda_3) - \frac{1}{2}\mu_2 - \frac{2}{3}\nu = 1 \quad \begin{array}{cccccc} 2 & 3 & 2 & 2 & 1 & 0 & 2 \end{array}$$

$$\frac{1}{2}(\lambda_0 - \lambda_1 - \lambda_2 - \lambda_3) - \frac{1}{2}\mu_2 + \frac{2}{3}\nu = 1 \quad 1 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1$$

$$\frac{1}{2}(\lambda_0 + \lambda_1 + \lambda_2 + \lambda_3) + \frac{1}{2}\mu_3 - \frac{2}{3}\nu = 1 \quad 2 \quad 3 \quad 3 \quad 3 \quad 2 \quad 1 \quad 1$$

$$\frac{1}{2}(\lambda_0 + \lambda_1 - \lambda_2 - \lambda_3) + \frac{1}{2}\mu_3 - \frac{2}{3}\nu = 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 0$$

$$\frac{1}{2}(\lambda_0 - \lambda_1 + \lambda_2 - \lambda_3) + \frac{1}{2}\mu_3 - \frac{2}{3}\nu = 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 0 \quad 0$$

$$\frac{1}{2}(\lambda_0 - \lambda_1 - \lambda_2 + \lambda_3) + \frac{1}{2}\mu_3 - \frac{2}{3}\nu = 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 0 \quad 0 \quad 0$$

$$\frac{1}{2}(\lambda_0 + \lambda_1 + \lambda_2 + \lambda_3) - \frac{1}{2}\mu_3 + \frac{2}{3}\nu = 1 \quad 3 \quad 5 \quad 4 \quad 3 \quad 2 \quad 1 \quad 3$$

$$\frac{1}{2}(\lambda_0 + \lambda_1 - \lambda_2 - \lambda_3) - \frac{1}{2}\mu_3 + \frac{2}{3}\nu = 1 \quad 2 \quad 3 \quad 2 \quad 1 \quad 1 \quad 1 \quad 2$$

$$\frac{1}{2}(\lambda_0 - \lambda_1 + \lambda_2 - \lambda_3) - \frac{1}{2}\mu_3 + \frac{2}{3}\nu = 1 \quad \begin{matrix} 2 & 3 & 2 & 1 & 1 & 0 & 2 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{matrix}$$

$$\frac{1}{2}(\lambda_0 - \lambda_1 - \lambda_2 + \lambda_3) - \frac{1}{2}\mu_3 + \frac{2}{3}\nu = 1 \quad 2 \quad 3 \quad 2 \quad 1 \quad 0 \quad 0 \quad 2$$

$$\begin{aligned}
\mu_1 - \frac{1}{3}\nu + r &= 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
\mu_2 - \frac{1}{3}\nu + r &= 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
-\mu_3 + \frac{1}{3}\nu + r &= 0 & 1 & 2 & 1 & 0 & 0 & 0 & 1 \\
\mu_1 - \frac{1}{3}\nu - r &= 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\mu_2 - \frac{1}{3}\nu + r &= 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
-\mu_3 - \frac{1}{3}\nu - r &= 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\
\\
\lambda_0 + \frac{1}{2}\mu_1 + \frac{1}{3}\nu + r &= 2 & 4 & 6 & 4 & 3 & 2 & 1 & 3 \\
\lambda_0 + \frac{1}{2}\mu_1 + \frac{1}{3}\nu - r &= 2 & 4 & 5 & 4 & 3 & 2 & 1 & 3 \\
\lambda_0 - \frac{1}{2}\mu_1 - \frac{1}{3}\nu + r &= 2 & 3 & 5 & 4 & 3 & 2 & 1 & 2 \\
\lambda_0 - \frac{1}{2}\mu_1 - \frac{1}{3}\nu - r &= 2 & 3 & 4 & 4 & 3 & 2 & 1 & 2 \\
\lambda_1 + \frac{1}{2}\mu_1 + \frac{1}{3}\nu + r &= 0 & 1 & 2 & 1 & 1 & 1 & 1 & 1 \\
\lambda_1 + \frac{1}{2}\mu_1 + \frac{1}{3}\nu - r &= 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\lambda_1 - \frac{1}{2}\mu_1 - \frac{1}{3}\nu + r &= 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \\
\lambda_1 - \frac{1}{2}\mu_1 - \frac{1}{3}\nu - r &= 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\
\lambda_2 + \frac{1}{2}\mu_1 + \frac{1}{3}\nu + r &= 0 & 1 & 2 & 1 & 1 & 1 & 0 & 1 \\
\lambda_2 + \frac{1}{2}\mu_1 + \frac{1}{3}\nu - r &= 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 \\
\lambda_2 - \frac{1}{2}\mu_1 - \frac{1}{3}\nu + r &= 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\
\lambda_2 - \frac{1}{2}\mu_1 - \frac{1}{3}\nu - r &= 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\
\lambda_3 + \frac{1}{2}\mu_1 + \frac{1}{3}\nu + r &= 0 & 1 & 2 & 1 & 1 & 0 & 0 & 1 \\
\lambda_3 + \frac{1}{2}\mu_1 + \frac{1}{3}\nu - r &= 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 \\
\lambda_3 - \frac{1}{2}\mu_1 - \frac{1}{3}\nu + r &= 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\
\lambda_3 - \frac{1}{2}\mu_1 - \frac{1}{3}\nu - r &= 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
\\
\frac{1}{2}(\lambda_0 + \lambda_1 + \lambda_2 - \lambda_3) + \frac{1}{2}\mu_2 + \frac{1}{3}\nu + r &= 1 & 2 & 4 & 3 & 2 & 2 & 1 & 2 \\
\frac{1}{2}(\lambda_0 + \lambda_1 + \lambda_2 - \lambda_3) + \frac{1}{2}\mu_2 + \frac{1}{3}\nu - r &= 1 & 2 & 3 & 3 & 2 & 2 & 1 & 2
\end{aligned}$$

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$$\begin{aligned}
\frac{1}{2}(\lambda_0 + \lambda_1 + \lambda_2 + \lambda_3) - \frac{1}{2}\mu_3 - \frac{1}{3}\nu + r &= 1 & 3 & 5 & 4 & 3 & 2 & 1 & 2 \\
\frac{1}{2}(\lambda_0 + \lambda_1 + \lambda_2 + \lambda_3) - \frac{1}{2}\mu_3 - \frac{1}{3}\nu - r &= 1 & 3 & 4 & 4 & 3 & 2 & 1 & 2 \\
\frac{1}{2}(\lambda_0 - \lambda_1 - \lambda_2 + \lambda_3) + \frac{1}{2}\mu_3 + \frac{1}{3}\nu + r &= 1 & 1 & 2 & 1 & 1 & 0 & 0 & 1 \\
\frac{1}{2}(\lambda_0 - \lambda_1 - \lambda_2 + \lambda_3) + \frac{1}{2}\mu_3 + \frac{1}{3}\nu - r &= 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 \\
\frac{1}{2}(\lambda_0 - \lambda_1 - \lambda_2 + \lambda_3) - \frac{1}{2}\mu_3 - \frac{1}{3}\nu + r &= 1 & 2 & 3 & 2 & 1 & 0 & 0 & 1 \\
\frac{1}{2}(\lambda_0 - \lambda_1 - \lambda_2 + \lambda_3) - \frac{1}{2}\mu_3 - \frac{1}{3}\nu - r &= 1 & 2 & 2 & 2 & 1 & 0 & 0 & 1 \\
\nu - r &= 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
2r &= 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
\nu + r &= 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1.
\end{aligned}$$

Hence $\Pi = \{\alpha_1, \alpha_2, \dots, \alpha_8\}$ is a fundamental root system of \mathfrak{e}_8^C . The real part $\mathfrak{h}_{\mathbf{R}}$ of \mathfrak{h} is

$$\mathfrak{h}_{\mathbf{R}} = \{(\Phi(h), 0, 0, r, 0) \mid \Phi(h) \in (\mathfrak{h}_7)_{\mathbf{R}}, r \in \mathbf{R}\},$$

(where $\Phi(h) = \{\Phi\left(\sum_{k=0}^3 \lambda_k H_k + \left(\sum_{j=1}^3 \mu_j E_j\right)^{\sim}, 0, 0, \nu\right) \in (\mathfrak{h}_7)_{\mathbf{R}}$ (Theorem 4.6.2)). The Killing form B_8 of \mathfrak{e}_8^C on $\mathfrak{h}_{\mathbf{R}}$ is given by

$$B_8(\tilde{h}, \tilde{h}') = 60 \sum_{k=0}^3 \lambda_k \lambda_k' + 30 \sum_{j=1}^3 \mu_j \mu_j' + 40\nu\nu' + 120rr',$$

for $\tilde{h} = (\Phi(h), 0, 0, r, 0)$, $\tilde{h}' = (\Phi(h'), 0, 0, r', 0) \in \mathfrak{h}$, where $\Phi(h) = \Phi\left(\sum_{k=0}^3 \lambda_k H_k + \left(\sum_{j=1}^3 \mu_j E_j\right)^{\sim}, 0, 0, \nu\right)$, $\Phi(h') = \Phi\left(\sum_{k=0}^3 \lambda_k' H_k + \left(\sum_{j=1}^3 \mu_j' E_j\right)^{\sim}, 0, 0, \nu'\right) \in (\mathfrak{h}_7)_{\mathbf{R}}$. Indeed,

$$\begin{aligned}
B_8(\tilde{h}, \tilde{h}') &= \frac{5}{3} B_7(\Phi(h), \Phi(h')) + 120rr' \quad (\text{Theorem 5.3.2}) \\
&= \frac{5}{3} 6 \left(6 \sum_{k=0}^3 \lambda_k \lambda_k' + 3 \sum_{j=1}^3 \mu_j \mu_j' + 4\nu\nu' \right) + 120rr' \quad (\text{Theorem 4.6.2}) \\
&= 60 \sum_{k=0}^3 \lambda_k \lambda_k' + 30 \sum_{j=1}^3 \mu_j \mu_j' + 40\nu\nu' + 120rr'.
\end{aligned}$$

Now, the canonical elements $H_{\alpha_i} \in \mathfrak{h}_{\mathbf{R}}$ associated with α_i ($B_8(H_{\alpha}, H) = \alpha(H)$, $H \in \mathfrak{h}_{\mathbf{R}}$) are determined as follows.

$$\begin{aligned}
H_{\alpha_1} &= \left(\Phi\left(\frac{1}{120}(H_0 - H_1 - H_2 - H_3) + 2(E_3 - E_1)^{\sim}, 0, 0, 0\right), 0, 0, 0, 0, 0\right), \\
H_{\alpha_2} &= \left(\Phi\left(\frac{1}{90}(2E_1 - E_2 - E_3)^{\sim}, 0, 0, -\frac{1}{120}\right), 0, 0, -\frac{1}{120}, 0, 0\right),
\end{aligned}$$

$$\begin{aligned}
H_{\alpha_3} &= \left(0, 0, 0, \frac{1}{60}, 0, 0\right), \\
H_{\alpha_4} &= \left(\Phi\left(\frac{1}{90}(-E_1 + 2E_2 - E_3)^\sim, 0, 0, -\frac{1}{120}\right), 0, 0, -\frac{1}{120}, 0, 0\right), \\
H_{\alpha_5} &= \left(\Phi\left(\frac{1}{60}(H_3 - (E_2 - E_3)^\sim), 0, 0, 0\right), 0, 0, 0, 0, 0\right), \\
H_{\alpha_6} &= \left(\Phi\left(\frac{1}{60}(H_2 - H_3), 0, 0, 0\right), 0, 0, 0, 0, 0\right), \\
H_{\alpha_7} &= \left(\Phi\left(\frac{1}{60}(H_1 - H_2), 0, 0, 0\right), 0, 0, 0, 0, 0\right), \\
H_{\alpha_8} &= \left(\Phi\left(0, 0, 0, \frac{1}{40}\right), 0, 0, -\frac{1}{120}, 0, 0\right).
\end{aligned}$$

Thus we have

$$(\alpha_1, \alpha_1) = B_7(H_{\alpha_1}, H_{\alpha_1}) = 60 \frac{1}{120} \frac{1}{120} 4 + 30 \frac{1}{120} \frac{1}{120} 8 = \frac{1}{30},$$

and the other inner products are similarly calculated. Consequently, the inner product induced by the Killing form B_8 between $\alpha_1, \alpha_2, \dots, \alpha_8$ and $-\mu$ are given by

$$\begin{aligned}
(\alpha_i, \alpha_i) &= \frac{1}{30}, \quad i = 1, 2, 3, 4, 5, 6, 7, 8, \\
(\alpha_i, \alpha_{i+1}) &= -\frac{1}{60}, \quad i = 1, 2, \dots, 6, \quad (\alpha_3, \alpha_8) = -\frac{1}{60}, \\
(\alpha_i, \alpha_j) &= 0, \quad \text{otherwise}, \\
(-\mu, -\mu) &= \frac{1}{30}, \quad (-\mu, \alpha_7) = -\frac{1}{60}, \quad (-\mu, \alpha_i) = 0, \quad i = 1, 2, 3, 4, 5, 6, 8,
\end{aligned}$$

using them, we can draw the Dynkin diagram and the extended Dynkin diagram of \mathfrak{e}_8^C .

According to Borel-Siebenthal theory, the Lie algebra \mathfrak{e}_8 has five subalgebras as maximal subalgebras with the maximal rank 8.

(1) The first one is a subalgebra of type $A_1 \oplus E_7$ which is obtained as the fixed points of an involution v of \mathfrak{e}_8 .

(2) The second one is a subalgebra of type D_8 which is obtained as the fixed points of an involution $\tilde{\lambda}_\gamma$ of \mathfrak{e}_8 .

(3) The third one is a subalgebra of type $A_2 \oplus E_6$ which is obtained as the fixed points of an automorphism w of order 3 of \mathfrak{e}_8 .

(4) The fourth one is a subalgebra of type A_8 which is obtained as the fixed points of an automorphism w_3 of order 3 of \mathfrak{e}_8 .

(5) The fifth one is a subalgebra of type $A_4 \oplus A_4$ which is obtained as the fixed points of an automorphism z_5 of order 5 of \mathfrak{e}_8 .

These subalgebras will be realized as subgroups of the group E_8 in Theorems 5.7.6, 5.8.7, 5.10.2, 5.11.7 and 5.12.5, respectively. As for Theorems 5.10.2, 5.11.7 and 5.12.5, we refer to Gomyo [9].

5.7. Involution v and subgroup $(SU(2) \times E_7)/Z_2$ of E_8

We shall first study the following subgroup $(E_8^C)_{1,1^-,1_-}$ of E_8^C :

$$(E_8^C)_{1,1^-,1_-} = \{\alpha \in E_8^C \mid \alpha 1 = 1, \alpha 1^- = 1^-, \alpha 1_- = 1_-\}.$$

Proposition 5.7.1. $(E_8^C)_{1,1^-,1_-} \cong E_7^C$.

Proof. For $\beta \in E_7^C$, we define a C -linear mapping $\tilde{\beta} : \mathfrak{e}_8^C \rightarrow \mathfrak{e}_8^C$ by

$$\tilde{\beta} = \begin{pmatrix} \text{Ad}\beta & 0 & 0 & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 & 0 & 0 \\ 0 & 0 & \beta & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

where $(\text{Ad}\beta)\Phi = \beta\Phi\beta^{-1}$, $\Phi \in \mathfrak{e}_7^C$. It is easy to check that $\tilde{\beta} \in (E_8^C)_{1,1^-,1_-}$. Conversely, if $\alpha \in E_8^C$ satisfies $\alpha 1 = 1, \alpha 1^- = 1^-$ and $\alpha 1_- = 1_-$, then α is of the form

$$\alpha = \begin{pmatrix} \beta_1 & \beta_{12} & \beta_{13} & 0 & 0 & 0 \\ \beta_{21} & \beta_2 & \beta_{23} & 0 & 0 & 0 \\ \beta_{31} & \beta_{32} & \beta_3 & 0 & 0 & 0 \\ a_1 & b_1 & c_1 & 1 & 0 & 0 \\ a_2 & b_2 & c_2 & 0 & 1 & 0 \\ a_3 & b_3 & c_3 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{array}{l} \beta_1 \in \text{Hom}_C(\mathfrak{e}_7^C), \\ \beta_2, \beta_3, \beta_{23}, \beta_{32} \in \text{Hom}_C(\mathfrak{P}^C), \\ \beta_{21}, \beta_{31} \in \text{Hom}_C(\mathfrak{e}_7^C, \mathfrak{P}^C), \\ \beta_{12}, \beta_{13} \in \text{Hom}_C(\mathfrak{P}^C, \mathfrak{e}_7^C), \\ a_i \in \text{Hom}_C(\mathfrak{e}_7^C, C), \\ b_i, c_i \in \text{Hom}_C(\mathfrak{P}^C, C). \end{array}$$

From the relation $[\alpha\Phi, 1] = [\alpha\Phi, \alpha 1] = \alpha[\Phi, 1] = 0$, that is,

$$\begin{aligned} 0 &= [(\beta_1\Phi, \beta_{21}\Phi, \beta_{31}\Phi, a_1\Phi, a_2\Phi, a_3\Phi), (0, 0, 0, 1, 0, 0)] \\ &= (0, -\beta_{21}\Phi, \beta_{31}\Phi, 0, -2a_2\Phi, 2a_3\Phi), \end{aligned}$$

we obtain $\beta_{21} = \beta_{31} = 0$ and $a_2 = a_3 = 0$. Furthermore, from $[\alpha\Phi, 1^-] = [\alpha\Phi, \alpha 1^-] = \alpha[\Phi, 1^-] = 0$, that is,

$$0 = [(\beta_1\Phi, 0, 0, a_1\Phi, 0, 0), (0, 0, 0, 0, 1, 0)] = (0, 0, 0, 0, 2a_1\Phi, 0),$$

we obtain $a_1 = 0$. Using the fact that $[\alpha P^-, 1] = [\alpha P^-, \alpha 1] = \alpha[P^-, 1] = -\alpha P^-$, that is,

$$\begin{aligned} &-(\beta_{12}P, \beta_2P, \beta_{32}P, b_1P, b_2P, b_3P) \\ &= [(\beta_{12}P, \beta_2P, \beta_{32}P, b_1P, b_2P, b_3P), (0, 0, 0, 1, 0, 0)] \\ &= (0, -\beta_2P, \beta_{32}P, 0, -2b_2P, 2b_3P), \end{aligned}$$

we obtain $\beta_{12} = \beta_{32} = 0$ and $b_1 = b_2 = b_3 = 0$. Similarly, from $[\alpha Q_-, 1] = [\alpha Q_-, \alpha 1] = \alpha[Q_-, 1] = \alpha Q_-$, we obtain $\beta_{13} = \beta_{23} = 0$ and $c_1 = c_2 = c_3 = 0$.

Thus we have seen that α is of the form

$$\alpha = \begin{pmatrix} \beta_1 & 0 & 0 & 0 \\ 0 & \beta_2 & 0 & 0 \\ 0 & 0 & \beta_3 & 0 \\ 0 & 0 & 0 & E \end{pmatrix}.$$

By applying α on $[(0, P, 0, 0, 0, 0), (0, 0, Q, 0, 0, 0)] = (P \times Q, 0, 0, -\frac{1}{8}\{P, Q\}, 0, 0)$, we obtain

$$\beta_1(P \times Q) = \beta_2 P \times \beta_3 Q, \quad \{\beta_2 P, \beta_3 Q\} = \{P, Q\}, \quad (\text{i})$$

since $[(0, \beta_2 P, 0, 0, 0, 0), (0, 0, \beta_3 Q, 0, 0, 0)] = (\beta_1(P \times Q), 0, 0, -\frac{1}{8}\{P, Q\}, 0, 0)$. Again, by applying α on $[(0, P, 0, 0, 0, 0), (0, Q, 0, 0, 0, 0)] = (0, 0, 0, \frac{1}{4}\{P, Q\}, 0, 0)$, we obtain

$$\{\beta_2 P, \beta_2 Q\} = \{P, Q\}. \quad (\text{ii})$$

Further, by applying α on $[(\Phi, 0, 0, 0, 0, 0), (0, P, 0, 0, 0, 0)] = (0, \Phi P, 0, 0, 0, 0)$, we obtain

$$(\beta_1 \Phi)(\beta_2 P) = \beta_2(\Phi P). \quad (\text{iii})$$

From (i), (ii), we have

$$\{\beta_2 P, \beta_3 Q\} = \{\beta_2 P, \beta_2 Q\}, \quad P, Q \in \mathfrak{P}^C,$$

hence $\beta_2 = \beta_3$, which we denote by β . If we put $\beta^{-1}P$ in (iii) in the place of P , we obtain

$$\beta_1 \Phi = \beta \Phi \beta^{-1}.$$

Therefore, from (i), we have

$$\beta(P \times Q)\beta^{-1} = \beta P \times \beta Q,$$

which implies that $\beta \in E_7^C$. Thus the proof of Theorem 5.4.1 is completed.

Definition. We define a C -linear mapping $v : \mathfrak{e}_8^C \rightarrow \mathfrak{e}_8^C$ by

$$v(\Phi, P, Q, r, s, t) = (\Phi, -P, -Q, r, s, t).$$

Then $v \in E_8$ and $v^2 = 1$. Note that v is the central element -1 of E_7 regarding as an element of E_8 (see Theorem 5.7.3).

We shall study the following subgroup $(E_8)^v$ of E_8 :

$$(E_8)^v = \{\alpha \in E_8 \mid v\alpha = \alpha v\}.$$

Lemma 5.7.2. *If $\alpha \in E_8$ satisfies $\alpha 1_- = 1_-$, then it also satisfies $\alpha 1 = 1$ and $\alpha 1^- = 1^-$.*

Proof. Let $\alpha 1 = (\Phi, P, Q, r, s, t)$. From the relation $[\alpha 1, 1_-] = [\alpha 1, \alpha 1_-] = \alpha[1, 1_-] = -2\alpha 1_- = -21_-$, we have

$$-21_- = [(\Phi, P, Q, r, s, t), 1_-] = (0, 0, -P, s, 0, -2r),$$

from which we obtain $P = 0, s = 0, r = 1$. Further, $\langle \alpha 1, \alpha 1 \rangle = \langle 1, 1 \rangle = 8$, that is, $\langle \Phi, \Phi \rangle + \langle Q, Q \rangle + 8 + 4(\tau t)t = 8$, which implies that $\Phi = 0, Q = 0, t = 0$. Therefore $\alpha 1 = 1$. By applying α on $[1^-, 1_-] = 1$, we obtain $\alpha 1^- = 1^-$ by a similar method to the above.

To find the structure of the group $(E_8)^v$, we shall first study the following subgroup $(E_8)_{1_-}$ of E_8 :

$$(E_8)_{1_-} = \{\alpha \in E_8 \mid \alpha 1_- = 1_-\}.$$

Theorem 5.7.3. $(E_8)_{1_-} \cong E_7$.

Proof. $(E_8)_{1_-} = (E_8)_{1, 1^-, 1_-}$ (Lemma 5.7.2)
 $= \{\alpha \in (E_8^C)_{1, 1^-, 1_-} \mid \tau \tilde{\lambda} \alpha = \alpha \tau \tilde{\lambda}\}$
 $= \{\alpha \in E_7^C \mid \tau \tilde{\lambda} \alpha = \alpha \tau \tilde{\lambda}\}$ (Proposition 5.7.1)
 $= \{\alpha \in E_7^C \mid \tau \lambda \alpha = \alpha \tau \lambda\}$ (by the correspondence to Proposition 5.7.1)
 $= E_7$ (Lemma 4.3.3.(4)).

Remark. We define a space \mathfrak{W}_1 by

$$\mathfrak{W}_1 = \{R \in \mathfrak{e}_8^C \mid R \times R = 0, \langle R, R \rangle = 4\}$$

(see Remark of Theorem 5.4.1), then, we obtain a homeomorphism

$$E_8/E_7 \simeq \mathfrak{W}_1.$$

(See Yokota, Imai and Yasukura [53]).

Theorem 5.7.4. *The group $(E_8)^v$ contains a subgroup*

$$\varphi_3(SU(2)) = \{\varphi_3(A) \in E_8 \mid A \in SU(2)\}$$

which is isomorphic to the group $SU(2) = \{A \in M(2, C) \mid (\tau^t A)A = E, \det A = 1\}$,

where, for $A = \begin{pmatrix} a & -\tau b \\ b & \tau a \end{pmatrix} \in SU(2)$, $\varphi_3(A) : \mathfrak{e}_8^C \rightarrow \mathfrak{e}_8^C$ is defined by

$$\varphi_3(A) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & a1 & -\tau b1 & 0 & 0 & 0 \\ 0 & b1 & \tau a1 & 0 & 0 & 0 \\ 0 & 0 & 0 & (\tau a)a - (\tau b)b & -(\tau a)b & a(\tau b) \\ 0 & 0 & 0 & 2a(\tau b) & a^2 & -(\tau b)^2 \\ 0 & 0 & 0 & 2(\tau a)b & -b^2 & (\tau a)^2 \end{pmatrix}.$$

Proof. For $A = \begin{pmatrix} a & -\tau b \\ b & \tau a \end{pmatrix} = \exp \begin{pmatrix} -i\nu & -\tau\rho \\ \rho & i\nu \end{pmatrix} \in SU(2)$, we have $\phi(A) = \exp(\Theta(0, 0, 0, i\nu, \rho, -\tau\rho)) \in (E_8)^v$.

Lemma 5.7.5. *The group $(E_8)^v$ is connected.*

Proof. $(E_8)^v$ is connected as a fixed points subgroup of the involutive automorphism v of the simply connected Lie group E_8 .

Theorem 5.7.6. $(E_8)^v \cong (SU(2) \times E_7)/\mathbf{Z}_2$, $\mathbf{Z}_2 = \{(E, 1), (-E, -1)\}$.

Proof. We define a mapping $\varphi : SU(2) \times E_7 \rightarrow (E_8)^v$ by

$$\varphi(A, \beta) = \varphi_3(A)\beta.$$

Since $\varphi_3(A) \in \varphi_3(SU(A))$ and $\beta \in E_7$ commute, φ is a homomorphism. Since $(E_8)^v$ is connected (Lemma 5.7.5), to prove that φ is onto, it is sufficient to show that the differential mapping $\varphi_* : \mathfrak{su}(8) \oplus \mathfrak{e}_7 \rightarrow (\mathfrak{e}_8)^v$ of φ is onto. But which is not difficult to see. Indeed, we have

$$\begin{aligned} (\mathfrak{e}_8)^v &= \{\Theta(R) \in \Theta(\mathfrak{e}_8) \mid v\Theta(R) = \Theta(R)v\} \cong \{R \in \mathfrak{e}_8 \mid vR = R\} \\ &= \{(\Phi, 0, 0, r, s, -\tau s) \mid \Phi \in \mathfrak{e}_7, r \in i\mathbf{R}, s \in C\}. \end{aligned}$$

$\text{Ker}\varphi = \{(E, 1), (-E, -1)\} = \mathbf{Z}_2$ is easily obtained. Thus we have the isomorphism $(SU(2) \times E_7)/\mathbf{Z}_2 \cong (E_8)^v$.

Remark. We can prove directly that φ is onto without using the connectedness of $(E_8)^v$ (Lemma 5.7.5), (see Imai and Yokota [13]).

5.8. Involution $\tilde{\lambda}\gamma$ and subgroup $Ss(16)$ of E_8

We define a C -linear mapping $\tilde{\lambda}\gamma : \mathfrak{e}_8^C \rightarrow \mathfrak{e}_8^C$ by

$$\tilde{\lambda}\gamma(\Phi, P, Q, , s, t) = (\lambda\gamma\Phi\gamma\lambda^{-1}, \lambda\gamma Q, -\lambda\gamma P, -r, -t, -s).$$

Then $\tilde{\lambda}\gamma \in E_8$ and $(\tilde{\lambda}\gamma)^2 = 1$.

We shall study the following subgroup $(E_8)^{\tilde{\lambda}\gamma}$ of E_8 :

$$\begin{aligned} (E_8)^{\tilde{\lambda}\gamma} &= \{\alpha \in E_8 \mid \tilde{\lambda}\gamma\alpha = \alpha\tilde{\lambda}\gamma\} \\ &= \{\alpha \in E_8 \mid \tau\gamma\alpha = \alpha\tau\gamma\} = (E_8)^{\tau\gamma}. \end{aligned}$$

We define an \mathbf{R} -linear mapping $l : M(8, C) \rightarrow M(16, \mathbf{R})$ by

$$l\left((x_{kl} + iy_{kl})\right) = \left(\begin{pmatrix} x_{kl} & y_{kl} \\ -y_{kl} & x_{kl} \end{pmatrix}\right), \quad x_{kl}, y_{kl} \in \mathbf{R}.$$

Further, we define $I, J \in M(16, \mathbf{R})$ by

$$I = \text{diag}(I, \dots, I), \quad I = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad J = \text{diag}(J, \dots, J), \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

then $IJ = -JI$, and for $X, Y \in M(8, C)$, we have

- (1) $l(XY) = l(X)l(Y)$,
- (2) $Il(X) = l(\tau X)I, \quad Jl(X) = l(X)J$,
- (3) ${}^tl(X) = l(\tau^t X)$.

Lemma 5.8.1. (1) $l(\mathfrak{u}(8)) = \{B \in \mathfrak{so}(16) \mid JB = BJ\}$,
 $l(\mathfrak{S}(8, C))I = \{B \in \mathfrak{so}(16) \mid JB = -BJ\}$.

(2) Any element $B \in \mathfrak{so}(16)$ is uniquely expressed by

$$\begin{aligned} B &= l(D') + l(S)I, & D' \in \mathfrak{u}(8), S \in \mathfrak{S}(8, C) \\ &= l(D) + l(S)I + l(icE), & D \in \mathfrak{su}(8), S \in \mathfrak{S}(8, C), c \in \mathbf{R}. \end{aligned}$$

Proof. (1) If $D \in \mathfrak{u}(8)$, then we have $Jl(D) = l(D)J$ and ${}^tl(D) = l(\tau^t D) = -l(D)$. Conversely, suppose that $B \in \mathfrak{so}(16)$ satisfies $JB = BJ$. Let $B = l(D)$, $D \in M(8, C)$. Then, the relation

$$l(-D) = -B = {}^tB = {}^tl(D) = l(\tau^t D)$$

implies $\tau^t D = -D$, that is, $D \in \mathfrak{u}(8)$. Next, for $S \in \mathfrak{S}(8, C)$, we have

$$\begin{aligned} Jl(S)I &= l(S)JI = -l(S)IJ, \\ {}^t(l(S)I) &= {}^tI {}^t(l(S)) = Il(\tau^t S) = -Il(\tau S) = -l(S)I. \end{aligned}$$

Conversely, suppose that $B \in \mathfrak{so}(16)$ satisfies $JB = -BJ$. Since the element BI satisfies $JB I = BI J$, we let $BI = l(S)$, $S \in M(8, C)$. Then the relation

$$l(-S)I = -B = {}^tB = {}^tI {}^t(l(S)) = Il(\tau^t S) = l({}^tS)I$$

implies $-S = {}^tS$, that is, $S \in \mathfrak{S}(8, C)$.

(2) Let $B = \frac{B - JBJ}{2} + \frac{B + JBJ}{2}$ and use (1) above.

The following Lemmas 5.8.2 and 5.8.3 are properties of the mappings $\chi : (\mathfrak{P}^C)_{\tau\gamma} \rightarrow \mathfrak{S}(8, C)$ and $\varphi_* : \mathfrak{su}(8) \rightarrow (\mathfrak{e}_6)^{\lambda\gamma}$ of Section 4.12, and will be used in the proof of Theorem 5.8.4.

Lemma 5.8.2 The Lie isomorphism $\varphi_* : \mathfrak{sp}(4) \rightarrow (\mathfrak{e}_6)^{\lambda\gamma}$ defined by $(\varphi_* D)X = g^{-1}(D(gX) + (gX)D^*)$, $X \in \mathfrak{J}^C$ of Theorem 3.1.2 satisfies

$$\varphi_*([gX_1, gX_2]) = 2(X_1 \vee \gamma X_2 - X_2 \vee \gamma X_1), \quad X_1, X_2 \in \mathfrak{J}^C.$$

Proof. Note that $[gX_1, gX_2] \in \mathfrak{sp}(4)$. Now, since

$$\begin{aligned} & g(2(X_1 \vee \gamma X_2)X) \quad X \in \mathfrak{J}^C \\ &= g((\gamma X_2, X)X_1 + \frac{1}{3}(X_1, \gamma X_2)X - 4\gamma X_2 \times (X_1 \times X)) \text{ (Lemma 3.4.1)} \\ &= (\gamma X_2, X)gX_1 + \frac{1}{3}(X_1, \gamma X_2)gX - 4gX_2 \circ (gX_1 \circ gX) + (\gamma X_1, X)gX_2 \\ &\quad + (\gamma X_2, \gamma X_1, \gamma X)E \text{ (Lemma 3.12.1),} \end{aligned}$$

we have

$$\begin{aligned} & g(2(X_1 \vee \gamma X_2 - X_2 \vee \gamma X_1)X) \\ &= -4gX_2 \circ (gX_1 \circ gX) + 4gX_1 \circ (gX_2 \circ gX) \\ &= -gX_2gX_1gX - gX_2gXgX_1 - gX_1gXgX_2 - gXgX_1gX_2 \\ &\quad + gX_1gX_2gX + gX_1gXgX_2 + gX_2gXgX_1 + gXgX_2gX_1 \\ &= [gX_1, gX_2]gX - gX[gX_1, gX_2] = g((\varphi_*[gX_1, gX_2])X). \end{aligned}$$

Consequently, since g is injective, we have the lemma.

Lemma 5.8.3. *For $S, S_1, S_2 \in \mathfrak{S}(8, C)$, we have*

- (1) $\lambda\gamma\chi^{-1}(S) = -\chi^{-1}(iS)$.
- (2) $\text{tr}(S_1\tau S_2 - S_2\tau S_1) = 4i\{\chi^{-1}S_1, \chi^{-1}S_2\}$.
- (3) $\varphi_*\left((S_1\tau S_2 - S_2\tau S_1) - \frac{1}{8}\text{tr}(S_1\tau S_2 - S_2\tau S_1)E\right) = 4(\lambda\gamma\chi^{-1}S_1 \times \chi^{-1}S_2 - \lambda\gamma\chi^{-1}S_2 \times \chi^{-1}S_1)$.

Proof. (1) Let $\chi^{-1}S = P = (X, Y, \xi, \eta)$. Then

$$\begin{aligned} \chi\lambda\gamma\chi^{-1}S &= \chi\lambda\gamma(X, Y, \xi, \eta) = \chi(\gamma Y, -\gamma X, \eta, -\xi) \\ &= \left(k\left(g(\gamma Y) - \frac{\eta}{2}E\right) + ik\left(g(\gamma(-\gamma X)) - \frac{-\xi}{2}E\right)\right)J \\ &= -i\left(k\left(gX - \frac{\xi}{2}E\right) + ik\left(g(\gamma Y) - \frac{\eta}{2}E\right)\right)J \\ &= -i\chi(X, Y, \xi, \eta) = -i\chi P = -iS. \end{aligned}$$

(2),(3) Let $\chi^{-1}S_i = P_i = (X_i, T_i, \xi_i, \eta_i), i = 1, 2$. Noting

$$S_1\tau S_2 - S_2\tau S_1 - \frac{1}{8}(S_1\tau S_2 - S_2\tau S_1) \in \mathfrak{su}(8),$$

we have

$$\begin{aligned} S_1\tau S_2 &= \chi P_1\tau\chi P_2 = \chi(X_1, Y_1, \xi_1, \eta_1)\tau\chi(X_2, Y_2, \xi_2, \eta_2) \\ &= \left(k\left(gX_1 - \frac{\xi_1}{2}E\right) + ik\left(g(\gamma Y_1) - \frac{\eta_1}{2}E\right)\right)J\tau\left(\left(k\left(gX_2 - \frac{\xi_2}{2}E\right) + ik\left(g(\gamma Y_2) - \frac{\eta_2}{2}E\right)\right)J\right) \end{aligned}$$

$$\begin{aligned}
&= \left(k \left(gX_1 - \frac{\xi_1}{2} E \right) + ik \left(g(\gamma Y_1) - \frac{\eta_1}{2} E \right) \right) \left(k \left(gX_2 - \frac{\xi_2}{2} E \right) \right. \\
&\quad \left. - ik \left(g(\gamma Y_2) - \frac{\eta_2}{2} E \right) \right) J^2 \\
&= -k \left(\left(gX_1 - \frac{\xi_1}{2} E \right) \left(gX_2 - \frac{\xi_2}{2} E \right) + \left(g(\gamma Y_1) - \frac{\eta_1}{2} E \right) \left(g(\gamma Y_2) - \frac{\eta_2}{2} E \right) \right) \\
&\quad - ik \left(\left(g(\gamma Y_1) - \frac{\eta_1}{2} E \right) \left(gX_2 - \frac{\xi_2}{2} E \right) - \left(gX_1 - \frac{\xi_1}{2} E \right) \left(g(\gamma Y_2) - \frac{\eta_2}{2} E \right) \right).
\end{aligned}$$

Therefore

$$\begin{aligned}
&S_1 \tau S_2 - S_2 \tau S_1 \\
&= -k(gX_1 gX_2 - gX_2 gX_1 + g(\gamma Y_1) g(\gamma Y_2) - g(\gamma Y_2) g(\gamma Y_1)) \\
&\quad - ik(g(\gamma Y_1) gX_2 - g(\gamma Y_2) gX_1 - gX_1 g(\gamma Y_2) + gX_2 g(\gamma Y_1)) \\
&\quad - \eta_1 gX_2 + \eta_2 gX_1 + \xi_1 g(\gamma Y_2) - \xi_2 g(\gamma Y_1) + \frac{i}{2}(\xi_1 \eta_2 - \eta_2 \xi_1) E \\
&= k(-[gX_1, gX_2] - [g(\gamma Y_1), g(\gamma Y_2)]) \\
&\quad + ik(g(2\gamma X_1 \times Y_2 - 2\gamma X_2 \times Y_1 + \eta_1 X_2 - \eta_2 X_1 - \xi_1 \gamma Y_2 + \xi_2 \gamma Y_1)) \\
&\quad + \frac{i}{2}((X_1, Y_2) - (X_2, Y_1) + \xi_1 \eta_2 - \xi_2 \eta_1) E \quad (\text{Lemma 3.12.1}) \\
&(\text{denote } D = -[gX_1, gX_2] - [g(\gamma Y_1), g(\gamma Y_2)] \in \mathfrak{sp}(4), \\
&\quad A = 2\gamma X_1 \times Y_2 - 2\gamma X_2 \times Y_1 + \eta_1 X_2 - \eta_2 X_1 - \xi_1 \gamma Y_2 + \xi_2 \gamma Y_1 \in \mathfrak{J}^C) \\
&= kD + ik(gA) + \frac{i}{2}\{P_1, P_2\}E.
\end{aligned}$$

Taking the trace of both sides, we obtain

$$\text{tr}(S_1 \tau S_2 - S_2 \tau S_1) = 4i\{P_1, P_2\} = 4i\{\chi^{-1}S_1, \chi^{-1}S_2\}.$$

and the expression above equal to

$$S_1 \tau S_2 - S_2 \tau S_1 - \frac{1}{8} \text{tr}(S_1 \tau S_2 - S_2 \tau S_1) = kD + ik(gA),$$

On the other hand, we have

$$\begin{aligned}
\lambda \gamma P_1 \times P_2 &= \begin{pmatrix} \gamma Y_1 \\ -\gamma X_1 \\ \eta_1 \\ -\xi_1 \end{pmatrix} \times \begin{pmatrix} X_2 \\ Y_2 \\ \xi_2 \\ \eta_2 \end{pmatrix} \\
&= \Phi \begin{pmatrix} -\frac{1}{2}(\gamma Y_1 \vee Y_2 - X_2 \vee \gamma X_1) \\ \frac{1}{4}(2\gamma X_1 \times Y_2 + \eta_1 X_2 + \xi_2 \gamma Y_1) \\ \frac{1}{4}(2\gamma Y_1 \times X_2 + \xi_1 Y_2 + \eta_2 \gamma X_1) \\ \frac{1}{8}((\gamma Y_1, Y_2) - (X_2, \gamma X_1) - 3(\eta_1 \eta_2 - \xi_2 \xi_1)) \end{pmatrix}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
& \lambda\gamma P_1 \times P_2 - \lambda\gamma P_2 \times P_1 \\
&= \Phi \begin{pmatrix} \frac{1}{2}(-X_1 \vee \gamma X_2 + X_2 \vee \gamma X_1 - \gamma Y_1 \vee Y_2 + \gamma Y_2 \vee Y_1) \\ \frac{1}{4}(2\gamma X_1 \times Y_2 - 2\gamma X_2 \times Y_1 + \eta_1 X_2 - \eta_2 X_1 + \xi_2 \gamma Y_1 - \xi_1 \gamma Y_2) \\ \frac{1}{4}(2X_2 \times \gamma Y_1 - 2X_1 \times \gamma Y_2 + \eta_2 \gamma X_1 - \eta_1 \gamma X_2 + \xi_1 Y_2 - \xi_2 Y_1) \\ 0 \end{pmatrix} \\
&= \Phi \left(\frac{1}{4} \varphi_* D, \frac{1}{4} A, -\frac{1}{4} \gamma A, 0 \right) \quad (\text{Lemma 5.8.2}) \\
&= \frac{1}{4} \varphi_* (kD + ik(gA)) \\
&= \frac{1}{4} \varphi_* \left(S_1 \tau S_2 - S_2 \tau S_1 - \frac{1}{8} \text{tr}(S_1 \tau S_2 - S_2 \tau S_1) E \right).
\end{aligned}$$

Theorem 5.8.4. *The Lie algebra $(\mathfrak{e}_8)^{\tilde{\lambda}\gamma}$ of the Lie group $(E_8)^{\tilde{\lambda}\gamma}$ is given by*

$$\begin{aligned}
(\mathfrak{e}_8)^{\tilde{\lambda}\gamma} &= \{ \Theta \in \Theta(\mathfrak{e}_8^C) \mid \tilde{\lambda}\gamma\Theta = \Theta\tilde{\lambda}\gamma \} \\
&= \{ \Theta(\Phi, P, -\lambda\gamma P, 0, s, -s) \mid \Phi \in (\mathfrak{e}_7)^{\lambda\gamma}, P \in (\mathfrak{P}^C)_{\lambda\gamma}, s \in \mathbf{R} \}
\end{aligned}$$

and $(\mathfrak{e}_8)^{\tilde{\lambda}\gamma}$ is isomorphic to the Lie algebra $\mathfrak{so}(16)$ by the mapping $\zeta : \mathfrak{so}(16) \rightarrow (\mathfrak{e}_8)^{\tilde{\lambda}\gamma}$ defined by

$$\zeta(l(D) + l(S)I + l(icE)) = \Theta(\varphi_* D, 2\lambda\gamma\chi^{-1}S, 2\chi^{-1}S, 0, 2c, -2c),$$

where $D \in \mathfrak{su}(8)$, $S \in \mathfrak{S}(8, C)$, $c \in \mathbf{R}$, and $\varphi_* : \mathfrak{su}(8) \rightarrow (\mathfrak{e}_7)^{\lambda\gamma}$, $\chi : (\mathfrak{P}^C)_{\text{tau}\gamma} \rightarrow \mathfrak{S}(8, C)$ are mappings defined in Section 4.12.

Proof. It is not difficult to see that the first half of the theorem and that ζ is onto. We will prove that ζ preserve the Lie bracket.

$$\begin{aligned}
(1) \quad & \zeta[l(D_1), l(D_2)] = \zeta l[D_1, D_2] \\
&= \Theta(\varphi_*[D_1, D_2], 0, 0, 0, 0, 0) = \Theta([\varphi_* D_1, \varphi_* D_2], 0, 0, 0, 0, 0) \\
&= [\Theta(\varphi_* D_1, 0, 0, 0, 0, 0), \Theta(\varphi_* D_2, 0, 0, 0, 0, 0)] \\
&= [\zeta l(D_1), \zeta l(D_2)]. \\
(2) \quad & \zeta[l(D), l(S)I] = \zeta(l(DS - S\tau D)I) = \zeta(l(DS + S^t D)I) \\
&= \Theta(0, 2\lambda\gamma\chi^{-1}(DS + S^t D), 2\chi^{-1}(DS + S^t D), 0, 0, 0).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
& [\zeta l(D), \zeta l(S)I] \\
&= [\Theta(\varphi_* D, 0, 0, 0, 0, 0), \Theta(0, 2\lambda\gamma\chi^{-1}S, 2\chi^{-1}S, 0, 0, 0)] \\
&= \Theta(0, 2(\varphi_* D)\lambda\gamma\chi^{-1}S, 2(\varphi_* D)\chi^{-1}S, 0, 0, 0).
\end{aligned}$$

Since $(\varphi_* D)\lambda\gamma = \lambda\gamma(\varphi_* D)$ and $(\varphi_* D)\chi^{-1}S = \chi^{-1}(DS + S^t D)$, they are equal.

$$\begin{aligned}
(3) \quad & \zeta[l(D), l(icE)] = \zeta[D, icE] = \zeta 0 = 0 \\
& = [\Theta(\varphi_* D, 0, 0, 0, 0, 0), \Theta(0, 0, 0, 0, 2c, -2c)] = [\zeta l(D), \zeta l(icE)]. \\
(4) \quad & \zeta[l(S_1)I, l(S_2)I] = \zeta l(S_1 \tau S_2 - S_2 \tau S_1) \\
& = \zeta \left(l \left(S_1 \tau S_2 - S_2 \tau S_1 - \frac{1}{8} \text{tr}(S_1 \tau S_2 - S_2 \tau S_1) E \right) \right. \\
& \quad \left. + l \left(\frac{1}{8} \text{tr}(S_1 \tau S_2 - S_2 \tau S_1) E \right) \right) \\
& = \Theta \left(\varphi_* \left(S_1 \tau S_2 - S_2 \tau S_1 - \frac{1}{8} \text{tr}(S_1 \tau S_2 - S_2 \tau S_1) E \right), 0, 0, 0, \right. \\
& \quad \left. - \frac{i}{4} \text{tr}(S_1 \tau S_2 - S_2 \tau S_1), \frac{i}{4} \text{tr}(S_1 \tau S_2 - S_2 \tau S_1) \right) \\
& = \Theta(4(\lambda \gamma \chi^{-1} S_1 \times \chi^{-1} S_2 - \lambda \gamma \chi^{-1} S_2 \times \chi^{-1} S_1), 0, 0, 0, \\
& \quad \{\chi^{-1} S_1, \chi^{-1} S_2\} - \{\chi^{-1} S_1, \chi^{-1} S_2\}) \quad (\text{Lemma 5.8.3}).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
& [\zeta l(S_1)I, \zeta l(S_2)I] \\
& = [\Theta(0, 2\lambda \gamma \chi^{-1} S_1, 2\chi^{-1} S_1, 0, 0, 0), \Theta(0, 2\lambda \gamma \chi^{-1} S_2, 2\chi^{-1} S_2, 0, 0, 0)] \\
& = \Theta \left(2\lambda \gamma \chi^{-1} S_1 \times 2\chi^{-1} S_2 - 2\lambda \gamma \chi^{-1} S_2 \times 2\chi^{-1} S_1, 0, 0, \right. \\
& \quad \left. \frac{1}{8}(-\{2\lambda \gamma \chi^{-1} S_1, 2\chi^{-1} S_2\} + \{2\lambda \gamma \chi^{-1} S_2, 2\chi^{-1} S_1\}), \right. \\
& \quad \left. \frac{1}{4}\{2\lambda \gamma \chi^{-1} S_1, 2\lambda \gamma \chi^{-1} S_2\}, -\frac{1}{4}\{2\chi^{-1} S_1, 2\chi^{-1} S_2\} \right),
\end{aligned}$$

which equals to the above.

$$\begin{aligned}
(5) \quad & \zeta[(icE), l(S)I] = \zeta(2l(icS)I) \\
& = \Theta(0, 4\lambda \gamma \chi^{-1}(icS), 4\chi^{-1}(icS), 0, 0, 0) \\
& = \Theta(0, 4\chi^{-1}(cS), -4\lambda \gamma \chi^{-1}(cS), 0, 0, 0) \quad (\text{Lemma 5.8.3}) \\
& = [\Theta(0, 0, 0, 0, 2c, -2c), \Theta(0, 2\lambda \gamma \chi^{-1} S, 2\chi^{-1} S, 0, 0, 0)] \\
& = [\zeta l(icE), \zeta l(S)I].
\end{aligned}$$

Finally,

$$\begin{aligned}
(6) \quad & \zeta[l(ic_1 E), l(ic_2 E)] = \zeta l[ic_1 E, ic_2 E] = \zeta 0 = 0 \\
& = \Theta(0, 0, 0, 0, 2c_1, -2c_1), \Theta(0, 0, 0, 0, 2c_2, -2c_2)) \\
& = [\zeta l(ic_1 E), \zeta l(ic_2 E)].
\end{aligned}$$

Thus we have proved Theorem 5.8.4.

The group $(E_8)^{\tilde{\lambda}\gamma}$ is connected as a fixed point subgroup under the involution $\tilde{\lambda}\gamma$ of the simply connected Lie group E_8 . Therefore, by Theorem 5.8.4, $(E_8)^{\tilde{\lambda}\gamma}$ is isomorphic to one of the following groups

$$Spin(16), \quad SO(16), \quad Ss(16), \quad SO(16)/\mathbf{Z}_2.$$

Precisely we have $(E_8)^{\tilde{\lambda}\gamma} \cong Ss(16)$, below we will give an outline of the proof.

We will use the Lie algebra

$$\mathfrak{e}_{8(8)} = (\mathfrak{e}_8^C)^{\tau\gamma} = \{R \in \mathfrak{e}_8^C \mid \tau\gamma R = R\}$$

(see Section 5.13). Now, consider the eigenspace decomposition of \mathfrak{e}_8^C by $\tilde{\lambda}\gamma$:

$$\mathfrak{e}_8^C = (\mathfrak{e}_8^C)_{\tilde{\lambda}\gamma} \oplus (\mathfrak{e}_8^C)_{-\tilde{\lambda}\gamma},$$

$$(\mathfrak{e}_8^C)_{\tilde{\lambda}\gamma} = \{\Theta \in \text{Der}(\mathfrak{e}_8^C) \mid \tilde{\lambda}\gamma\Theta = \Theta\tilde{\lambda}\gamma\} \cong \{R \in \mathfrak{e}_8^C \mid \tilde{\lambda}\gamma R = R\} = (\mathfrak{e}_8^C)^{\tilde{\lambda}\gamma},$$

$$(\mathfrak{e}_8^C)_{-\tilde{\lambda}\gamma} = \{\Theta \in \text{Der}(\mathfrak{e}_8^C) \mid \tilde{\lambda}\gamma\Theta = -\Theta\tilde{\lambda}\gamma\} \cong \{R \in \mathfrak{e}_8^C \mid \tilde{\lambda}\gamma R = -R\},$$

Since we have

$$\begin{aligned} ((\mathfrak{e}_8^C)_{\tilde{\lambda}\gamma})_{\tau\tilde{\lambda}} &= ((\mathfrak{e}_8^C)_{\tau\tilde{\lambda}})_{\tilde{\lambda}\gamma} = (\mathfrak{e}_{8(8)})_{\tilde{\lambda}\gamma} = (\mathfrak{e}_{8(8)})^{\tilde{\lambda}\gamma}, \\ ((\mathfrak{e}_8^C)_{-\tilde{\lambda}\gamma})_{\tau\tilde{\lambda}} &= ((\mathfrak{e}_8^C)_{\tau\tilde{\lambda}})_{-\tilde{\lambda}\gamma} = (\mathfrak{e}_{8(8)})_{-\tilde{\lambda}\gamma}, \end{aligned}$$

we obtain the following decomposition of $\mathfrak{e}_{8(8)}$:

$$\mathfrak{e}_{8(8)} = (\mathfrak{e}_{8(8)})^{\tilde{\lambda}\gamma} \oplus (\mathfrak{e}_{8(8)})_{-\tilde{\lambda}\gamma}.$$

Since $(\mathfrak{e}_{8(8)})^{\tilde{\lambda}\gamma} \cong \text{Spin}(16)$ (Theorem 5.8.4), this is the Cartan decomposition of $\mathfrak{e}_{8(8)}$. Since $[(\mathfrak{e}_{8(8)})^{\tilde{\lambda}\gamma}, (\mathfrak{e}_{8(8)})_{-\tilde{\lambda}\gamma}] \subset (\mathfrak{e}_{8(8)})_{-\tilde{\lambda}\gamma}$, we obtain a representation φ of $(\mathfrak{e}_{8(8)})^{\tilde{\lambda}\gamma}$ to $(\mathfrak{e}_{8(8)})_{-\tilde{\lambda}\gamma}$:

$$\varphi(R)R_1 = [R, R_1], \quad R \in (\mathfrak{e}_{8(8)})^{\tilde{\lambda}\gamma}, R_1 \in (\mathfrak{e}_{8(8)})_{-\tilde{\lambda}\gamma}.$$

which is irreducible. (See, for example, (8.5.1) of Goto and Grosshans [11]). Furthermore, the complex representation φ^C of φ to $((\mathfrak{e}_{8(8)})_{-\tilde{\lambda}\gamma})^C = (\mathfrak{e}_8^C)_{-\tilde{\lambda}\gamma}$ is also irreducible, since $(\mathfrak{e}_8^C)_{-\tilde{\lambda}\gamma}$ is simple (see (8.8.3) of the same book). The following lemma follows from above mentioned results.

Lemma 5.8.5. *The representation of the group $(E_8)^{\tilde{\lambda}\gamma}$ to $(\mathfrak{e}_{8(8)})_{-\tilde{\lambda}\gamma}$ is irreducible.*

Proposition 5.8.6. *The center $z((E_8)^{\tilde{\lambda}\gamma})$ of the group $(E_8)^{\tilde{\lambda}\gamma}$ is a group of order 2:*

$$z((E_8)^{\tilde{\lambda}\gamma}) = \{1, \tilde{\lambda}\gamma\}.$$

Proof. Evidently, $\{1, \tilde{\lambda}\gamma\} \subset z((E_8)^{\tilde{\lambda}\gamma})$. Conversely, let $\alpha \in z((E_8)^{\tilde{\lambda}\gamma})$. Since the representation of $(E_8)^{\tilde{\lambda}\gamma}$ to $(\mathfrak{e}_8^C)_{-\tilde{\lambda}\gamma}$ is irreducible (Lemma 5.8.5), we see, by using Schur's lemma in the theory of groups, that the action of α on $(\mathfrak{e}_8^C)_{-\tilde{\lambda}\gamma}$ is constant. Therefore, there exists an element $k \in C$ such that

$$\alpha R = kR, \quad R \in (\mathfrak{e}_8^C)_{-\tilde{\lambda}\gamma}.$$

Since the Killing form $B_8(R, R')$ is invariant under α : $B_8(\alpha R, \alpha R') = B_8(R, R')$, we have

$$k^2 B_8(R, R') = B_8(\alpha R, \alpha R') = B_8(R, R'), \quad R, R' \in (\mathfrak{e}_8^C)_{-\tilde{\lambda}\gamma},$$

which implies that $k^2 = 1$. By Theorem 5.8.4, we have $(\mathfrak{e}_8^C)^{\tilde{\lambda}\gamma} \cong \mathfrak{so}(16, C)$ which is simple, and hence we see that $(\mathfrak{e}_8^C)^{\tilde{\lambda}\gamma}$ is generated by $(\mathfrak{e}_8^C)^{-\tilde{\lambda}\gamma}$:

$$(\mathfrak{e}_8^C)^{\tilde{\lambda}\gamma} = \left\{ \sum_{k,l} [R_k, R_l] \mid R_k, R_l \in (\mathfrak{e}_8^C)_{-\tilde{\lambda}\gamma} \right\}.$$

Consequently, α satisfies $k^2 1 = 1$ on $(\mathfrak{e}_8^C)_{-\tilde{\lambda}\gamma}$, that is, the identity mapping. When $k = 1$, we have $\alpha = 1$, when $k = -1$, we have $\alpha = \tilde{\lambda}\gamma$. Thus we have proved the theorem.

It follows from Proposition 5.8.6, that $(E_8)^{\tilde{\lambda}\gamma}$ is isomorphic to one of the following

$$SO(16), \quad Ss(16).$$

There are only two, up to equivalence, complex irreducible representations of the Lie algebra $\mathfrak{so}(16)$ of dimension 128. In fact, one can obtain the following table, by calculating the dimension of dominant roots by virtue of Weyl's dimension formula (see (7.5.9) of Goto and Grosshans [11]):

ω_1	$2\omega_1$	ω_2	$2\omega_2$	ω_3	ω_4	ω_5	ω_6	ω_7	ω_8	\cdots
16	135	120	5304	560	1820	4368	8008	128	128	\cdots

hence the dominant root of dimension 128 is either ω_7 or ω_8 . (Here, $\omega_1, \omega_2, \dots, \omega_8$ are fundamental weights). On the other hand, $Spin(16)$ has two complex irreducible representation Δ_{16}^+ and Δ_{16}^- , called spinor representations. Furthermore, both of Δ_{16}^+ and Δ_{16}^- are not representation of $SO(16)$. Now, by Lemma 5.8.5, $(E_8)^{\tilde{\lambda}\gamma}$ has a complex irreducible representation $(\mathfrak{e}_8^C)^{-\tilde{\lambda}\gamma}$ of dimension 128, which implies that $(E_8)^{\tilde{\lambda}\gamma}$ is not $SO(16)$. So $(E_8)^{\tilde{\lambda}\gamma}$ must be $Ss(16)$. Thus we have proved the following theorem.

Theorem 5.8.7. $(E_8)^{\tilde{\lambda}\gamma} \cong Ss(16)$.

Remark. We define an involution C -linear transformation σ of \mathfrak{e}_8 by

$$\sigma(\Phi, P, Q, r, s, t) = (\sigma\Phi\sigma, \sigma P, \sigma Q, r, s, t).$$

This is the same as $\sigma \in F_4 \subset E_6 \subset E_7 \subset E_8$. We define a subgroup $(E_8)^\sigma$ by

$$(E_8)^\sigma = \{\alpha \in E_8 \mid \sigma\alpha = \alpha\sigma\}.$$

Then we have

$$(E_8)^\sigma \cong Ss(16).$$

Indeed, we can prove that the Lie algebra $(\mathfrak{e}_8)^\sigma$ of the group $(E_8)^\sigma$ is isomorphic to the Lie algebra $\mathfrak{so}(16) = \{X \in M(16, \mathbf{R}) \mid {}^t X = -X\}$. Hence as the same the case the group $(E_8)^{\tilde{\lambda}\gamma}$, the group $(E_8)^\sigma$ have to the semi-spinor group $Ss(16)$. However the proof of $(E_8)^{\tilde{\lambda}\gamma} \cong Ss(16) \cong (E_8)^\sigma$ is not concretely. M. Gomyo [10] find the group

$Ss(16)$ explicitly in the group E_8 (although the definition of the group E_8 is different from E_8 in Section 5.5).

5.9. Center $z(E_8)$ of E_8

Theorem 5.9.1. *The center $z(E_8)$ of the group E_8 is trivial:*

$$z(E_8) = \{1\}.$$

Proof. Let $\alpha \in z(E_8)$. The relation $v\alpha = \alpha v$ implies that $\alpha \in \varphi(SU(2) \times E_7) \cong (SU(2) \times E_7)/Z_2$ (Theorem 5.7.6), and so $\alpha \in z(\varphi(SU(2) \times E_7))$. Therefore by Theorem 4.9.1 we have

$$\alpha = \varphi(E, 1) = 1 \quad \text{or} \quad \alpha = \varphi(E, -1) = v.$$

However $v \notin z(E_8)$ (Theorem 5.7.6). Hence $\alpha = 1$.

5.10. Automorphism w of order 3 and subgroup $(SU(3) \times E_6)/Z_3$ of E_8

In this section (also in the following Sections 5.11 and 5.12), we use the same notation as $\mathfrak{e}_8^C, \langle R_1, R_2 \rangle, \tau\tilde{\lambda}, w$, even if these are different from those used in the proceeding sections.

We consider a $27 \times 3 = 78$ dimensional C -vector space

$$(\mathfrak{J}^C)^3 = \left\{ \mathbf{X} = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} \mid X_i \in \mathfrak{J}^C \right\}.$$

In $(\mathfrak{J}^C)^3$, we define an inner product (\mathbf{X}, \mathbf{Y}) , a Hermitian inner product $\langle \mathbf{X}, \mathbf{Y} \rangle$, a cross product $\mathbf{X} \times \mathbf{Y}$, an element $\mathbf{X} \cdot \mathbf{Y}$ of $\mathfrak{sl}(3, C)$ and an element $\mathbf{X} \vee \mathbf{Y}$ of \mathfrak{e}_6^C respectively by

$$(\mathbf{X}, \mathbf{Y}) = (X_1, Y_1) + (X_2, Y_2) + (X_3, Y_3) \in C,$$

$$\langle \mathbf{X}, \mathbf{Y} \rangle = \langle X_1, Y_1 \rangle + \langle X_2, Y_2 \rangle + \langle X_3, Y_3 \rangle \in C,$$

$$\mathbf{X} \times \mathbf{Y} = \begin{pmatrix} X_2 \times Y_3 - Y_2 \times X_3 \\ X_3 \times Y_1 - Y_3 \times X_1 \\ X_1 \times Y_2 - Y_1 \times X_2 \end{pmatrix} \in (\mathfrak{J}^C)^3,$$

$$\mathbf{X} \cdot \mathbf{Y} = \begin{pmatrix} (X_1, Y_1) & (X_1, Y_2) & (X_1, Y_3) \\ (X_2, Y_1) & (X_2, Y_2) & (X_2, Y_3) \\ (X_3, Y_1) & (X_3, Y_2) & (X_3, Y_3) \end{pmatrix} - \frac{1}{3}(\mathbf{X}, \mathbf{Y})E \in \mathfrak{sl}(3, C),$$

$$\mathbf{X} \vee \mathbf{Y} = X_1 \vee Y_1 + X_2 \vee Y_2 + X_3 \vee Y_3 \in \mathfrak{e}_6^C,$$

where $\mathbf{X} = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}, \mathbf{Y} = \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix} \in (\mathfrak{J}^C)^3$. Further, for $\phi \in \text{Hom}_C(\mathfrak{J}^C), D = (d_{ij}) \in$

$M(3, C)$ and $\mathbf{X} = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} \in (\mathfrak{J}^C)^3$, elements $\phi\mathbf{X}, D\mathbf{X} \in (\mathfrak{J}^C)^3$ are naturally defined

by

$$\phi \mathbf{X} = \begin{pmatrix} \phi X_1 \\ \phi X_2 \\ \phi X_3 \end{pmatrix}, \quad D \mathbf{X} = \begin{pmatrix} d_{11}X_1 + d_{12}X_2 + d_{13}X_3 \\ d_{21}X_1 + d_{22}X_2 + d_{23}X_3 \\ d_{31}X_1 + d_{32}X_2 + d_{33}X_3 \end{pmatrix}.$$

Theorem 5.10.1. *In an $8 + 78 + 27 \times 3 + 27 \times 3 = 248$ dimensional C -vector space*

$$\mathfrak{e}_8^C = \mathfrak{sl}(3, C) \oplus \mathfrak{e}_6^C \oplus (\mathfrak{J}^C)^3 \oplus (\mathfrak{J}^C)^3,$$

we define a Lie bracket $[R_1, R_2]$ by

$$[(D_1, \phi_1, \mathbf{X}_1, \mathbf{Y}_1), (D_2, \phi_2, \mathbf{X}_2, \mathbf{Y}_2)] = (D, \phi, \mathbf{X}, \mathbf{Y}),$$

where

$$\begin{cases} D = [D_1, D_2] + \frac{1}{4}\mathbf{X}_1 \cdot \mathbf{Y}_2 - \frac{1}{4}\mathbf{X}_2 \cdot \mathbf{Y}_1 \\ \phi = [\phi_1, \phi_2] + \frac{1}{2}\mathbf{X}_1 \vee \mathbf{Y}_2 - \frac{1}{2}\mathbf{X}_2 \vee \mathbf{Y}_1 \\ \mathbf{X} = \phi_1 \mathbf{X}_2 - \phi_2 \mathbf{X}_1 + D_1 \mathbf{X}_2 - D_2 \mathbf{X}_1 - \mathbf{Y}_1 \times \mathbf{Y}_2 \\ \mathbf{Y} = -{}^t\phi_1 \mathbf{Y}_2 + {}^t\phi_2 \mathbf{Y}_1 - {}^tD_1 \mathbf{Y}_2 + {}^tD_2 \mathbf{Y}_1 + \mathbf{X}_1 \times \mathbf{X}_2, \end{cases}$$

then \mathfrak{e}_8^C becomes a C -Lie algebra of type E_8 .

Proof. Let $\tilde{\mathfrak{e}}_8^C = \mathfrak{e}_7^C \oplus \mathfrak{P}^C \oplus \mathfrak{P}^C \oplus C \oplus C \oplus C$ be the C -Lie algebra constructed in Theorem 5.1.1. We define a mapping $f : \tilde{\mathfrak{e}}_8^C \rightarrow \mathfrak{e}_8^C$ by

$$\begin{aligned} & f(\Phi(\phi, A, B, \nu), (X, Y, \xi, \eta), (Z, W, \zeta, \omega), r, s, t)) \\ &= \left(\begin{pmatrix} \frac{2}{3}\nu & -\frac{1}{2}\xi & \frac{1}{2}\zeta \\ \frac{1}{2}\omega & -\frac{1}{3}\nu - r & t \\ \frac{1}{2}\eta & s & -\frac{1}{3}\nu + r \end{pmatrix}, \phi, \begin{pmatrix} -2A \\ Z \\ X \end{pmatrix}, \begin{pmatrix} -2B \\ Y \\ -W \end{pmatrix} \right), \end{aligned}$$

then we can prove that f is an isomorphism as Lie algebras by straightforward calculations. Thus we have the isomorphism $\tilde{\mathfrak{e}}_8^C \cong \mathfrak{e}_8^C$.

Using the Killing form of $\tilde{\mathfrak{e}}_8^C$ which is obtained in Theorem 5.3.2, we see that the Killing form B_8 of \mathfrak{e}_8^C is given by

$$B_8(R_1, R_2) = 60\text{tr}(D_1 D_2) + \frac{5}{2}B_6(\phi_1, \phi_2) + 15(\mathbf{X}_1, \mathbf{Y}_2) + 15(\mathbf{X}_2, \mathbf{Y}_1)$$

($R_i = (D_i, \phi_i, \mathbf{X}_i, \mathbf{Y}_i) \in \mathfrak{e}_8^C$), where B_6 is the Killing form of \mathfrak{e}_6^C . We define a complex conjugate transformation $\tau\tilde{\lambda}$ of \mathfrak{e}_8^C by

$$\tau\tilde{\lambda}(D, \phi, \mathbf{X}, \mathbf{Y}) = (-\tau^t D, -\tau^t \phi \tau, -\tau \mathbf{Y}, -\tau \mathbf{X}).$$

And we define a Hermitian inner product $\langle R_1, R_2 \rangle$ in \mathfrak{e}_8^C by

$$\langle R_1, R_2 \rangle = -B_8(R_1, \tau\tilde{\lambda}R_2).$$

Then we have

$$\langle R_1, R_2 \rangle = 60\text{tr}(D_1(\tau^t D_2)) + \frac{5}{2}B_6(\phi_1, \tau^t \phi_2 \tau) + 15\langle \mathbf{X}_1, \mathbf{X}_2 \rangle + 15\langle \mathbf{Y}_1, \mathbf{Y}_2 \rangle.$$

Now, as in Theorem 5.5.3, we see that

$$E_8 = \{\alpha \in \text{Aut}(\mathfrak{e}_8^C) \mid \langle \alpha R_1, \alpha R_2 \rangle = \langle R_1, R_2 \rangle\}$$

is a simply connected compact Lie group of type E_8 .

We define a C -linear transformation w of \mathfrak{e}_8^C by

$$w(D, \phi, \mathbf{X}, \mathbf{Y}) = (D, \phi, \omega \mathbf{X}, \omega^2 \mathbf{Y}),$$

where $\omega = -\frac{1}{2} + \frac{\sqrt{3}}{2}i \in C$. Then $w \in E_8$ and $w^3 = 1$.

Now, we shall study the following subgroup $(E_8)^w$ of E_8 :

$$(E_8)^w = \{\alpha \in E_8 \mid w\alpha = \alpha w\}.$$

Theorem 5.10.2. $(E_8)^w \cong (SU(3) \times E_6)/\mathbf{Z}_3$, $\mathbf{Z}_3 = \{(E, 1), (\omega E, \omega^2 1), (\omega^2 E, \omega 1)\}$.

Proof. We first define a mapping $\varphi_1 : SU(3) \rightarrow (E_8)^w$ by

$$\varphi_1(A)(D, \phi, \mathbf{X}, \mathbf{Y}) = (ADA^{-1}, \phi, A\mathbf{X}, {}^t A^{-1} \mathbf{Y}).$$

We have to prove that $\varphi_1(A) \in (E_8)^w$. Indeed, since the action of $D_1 = (D_1, 0, 0, 0) \in \mathfrak{su}(3) \subset \mathfrak{sl}(3, C) \subset \mathfrak{e}_8^C$ is given by

$$(\text{ad} D_1)(D, \phi, \mathbf{X}, \mathbf{Y}) = ((\text{ad} D_1)D, 0, D_1 \mathbf{X}, -{}^t D_1 \mathbf{Y}),$$

for $A = \exp D_1$, we have $\varphi_1(A) = \exp(\text{ad}(D_1)) \in \text{Aut}(\mathfrak{e}_8^C)$. And from

$$\begin{aligned} \text{tr}(AD_1 \tau^t A (\tau^t (AD_2 \tau^t A))) &= \text{tr}(AD_1 (\tau^t D_2) A^{-1}) = \text{tr}(D_1 (\tau^t D_2)), \\ \langle A\mathbf{X}, A\mathbf{Y} \rangle &= \langle \mathbf{X}, \mathbf{Y} \rangle, \end{aligned}$$

we see that $\varphi_1(A) \in E_8$. Evidently, $w\varphi_1(A) = \varphi_1(A)w$, hence, $\varphi_1(A) \in (E_8)^w$. Next, we define a mapping $\varphi_2 : E_6 \rightarrow (E_8)^w$ by

$$\varphi_2(\alpha)(D, \phi, \mathbf{X}, \mathbf{Y}) = (D, \alpha \phi \alpha^{-1}, \alpha \mathbf{X}, {}^t \alpha^{-1} \mathbf{Y}).$$

We have to prove that $\varphi_2(\alpha) \in (E_8)^w$. Indeed, since the action of an element $\phi' = (0, \phi', 0, 0) \in \mathfrak{e}_6 \subset \mathfrak{e}_8^C$ is given by

$$(\text{ad} \phi')(D, \phi, \mathbf{X}, \mathbf{Y}) = (0, (\text{ad} \phi')\phi, \phi' \mathbf{X}, -{}^t \phi' \mathbf{Y}),$$

for $\alpha = \exp \phi'$, we have $\varphi_2(\alpha) = \exp(\text{ad}(\phi')) \in \text{Aut}(\mathfrak{e}_8^C)$. And from

$$B_6(\alpha\phi_1\alpha^{-1}, \tau^t(\alpha\phi_2\alpha^{-1})\tau) = B_6(\alpha\phi_1\alpha^{-1}, \alpha\tau^t\phi_2\tau\alpha^{-1}) = B_6(\phi_1, \tau^t\phi_2\tau),$$

$$\langle \alpha \mathbf{X}, {}^t\alpha^{-1}\mathbf{Y} \rangle = \langle \mathbf{X}, \mathbf{Y} \rangle,$$

we see that $\varphi_2(\alpha) \in E_8$. Evidently, $w\varphi_2(\alpha) = \varphi_2(\alpha)w$, hence, $\varphi_2(\alpha) \in (E_8)^w$. Now, we define a mapping $\varphi : SU(3) \times E_6 \rightarrow (E_8)^w$ by

$$\varphi(A, \alpha) = \varphi_1(A)\varphi_2(\alpha).$$

Since $\varphi_1(A)$ and $\varphi_2(\alpha)$ commute, φ is a homomorphism. It is not difficult to show that $\text{Ker}\varphi = \{(E, 1), (\omega E, \omega^2 1), (\omega^2 E, \omega 1)\} = \mathbf{Z}_3$. Finally, since $(E_8)^w$ is connected as the fixed points subgroup by automorphisms w of the simply connected group E_8 and

$$\begin{aligned} (\mathfrak{e}_8)^w &= \{R \in \mathfrak{e}_8^C \mid wR = R, \tau\tilde{\lambda}R = R\} \\ &= \{(D, \phi, 0, 0) \in \mathfrak{e}_8^C \mid D \in \mathfrak{su}(3), \phi \in \mathfrak{e}_6\} \cong \mathfrak{su}(3) \oplus \mathfrak{e}_6, \end{aligned}$$

φ is onto. Thus we have the isomorphism $(SU(3) \times E_6)/\mathbf{Z}_3 \cong (E_8)^w$.

5.11. Automorphism w_3 of order 3 and subgroup $SU(9)/\mathbf{Z}_3$ of E_8

In order to construct another C -Lie algebra of type E_8 , we investigate the properties of the exterior C -vector space $\Lambda^k(C^n)$. Let $\mathbf{e}_1, \dots, \mathbf{e}_n$ be the canonical C -basis of n -dimensional C -vector space C^n and (\mathbf{x}, \mathbf{y}) the inner product in C^n satisfying $(\mathbf{e}_i, \mathbf{e}_j) = \delta_{ij}$. In $\Lambda^k(C^n)$, we define an inner product by

$$\begin{aligned} (\mathbf{x}_1 \wedge \dots \wedge \mathbf{x}_k, \mathbf{y}_1 \wedge \dots \wedge \mathbf{y}_k) &= \det\left((\mathbf{x}_i, \mathbf{y}_j)\right), \quad k \geq 1, \\ (a, b) &= ab, \quad a, b \in \Lambda^0(C^n) = C. \end{aligned}$$

Then, $\mathbf{e}_{i_1} \wedge \dots \wedge \mathbf{e}_{i_k}, i_1 < \dots < i_k$ forms an orthonormal C -basis of $\Lambda^k(C^n)$. For $\mathbf{u} \in \Lambda^k(C^n)$, we define an element $*\mathbf{u} \in \Lambda^{n-k}(C^n)$ by

$$(*\mathbf{u}, \mathbf{v}) = (\mathbf{u} \wedge \mathbf{v}, \mathbf{e}_1 \wedge \dots \wedge \mathbf{e}_n), \quad \mathbf{v} \in \Lambda^{n-k}(C^n).$$

Then, $*$ induces a C -linear isomorphism

$$* : \Lambda^k(C^n) \rightarrow \Lambda^{n-k}(C^n)$$

and satisfies the following identity:

$$*^2\mathbf{u} = (-1)^{k(n-k)}\mathbf{u}, \quad \mathbf{u} \in \Lambda^k(C^n).$$

The group $SL(n, C)$ naturally acts on $\Lambda^k(C^n)$ as

$$A(\mathbf{x}_1 \wedge \dots \wedge \mathbf{x}_k) = A\mathbf{x}_1 \wedge \dots \wedge A\mathbf{x}_k, \quad A1 = 1.$$

Hence the Lie algebra $\mathfrak{sl}(n, C)$ acts on $\Lambda^k(C^n)$ as

$$D(x_1 \wedge \cdots \wedge x_k) = \sum_{j=1}^k x_1 \wedge \cdots \wedge Dx_j \wedge \cdots \wedge x_k, \quad D1 = 0.$$

Lemma 5.11.1. For $A \in SL(n, C)$, $D \in \mathfrak{sl}(n, C)$ and $\mathbf{u}, \mathbf{v} \in \Lambda^k(C^n)$, we have

- (1) $(A\mathbf{u}, {}^tA^{-1}\mathbf{v}) = (\mathbf{u}, \mathbf{v}), \quad (D\mathbf{u}, \mathbf{v}) + (\mathbf{u}, -{}^tD\mathbf{v}) = 0.$
- (2) $*(A\mathbf{u}) = {}^tA^{-1}(*\mathbf{u}), \quad *(D\mathbf{u}) = -{}^tD^{-1}(*\mathbf{u}).$

For $\mathbf{u}, \mathbf{v} \in \Lambda^k(C^n)$ ($1 \leq k \leq n$), we define a C -linear mapping $\mathbf{u} \times \mathbf{v}$ of C^n by

$$(\mathbf{u} \times \mathbf{v})\mathbf{x} = *(\mathbf{v} \wedge *(\mathbf{u} \wedge \mathbf{x})) + (-1)^{n-k} \frac{n-k}{n} (\mathbf{u}, \mathbf{v})\mathbf{x}, \quad \mathbf{x} \in C^n.$$

Since $\text{tr}(\mathbf{u} \times \mathbf{v}) = 0$, $\mathbf{u} \times \mathbf{v}$ can be regarded as element of $\mathfrak{sl}(n, C)$ with respect to the canonical basis of C^n .

Lemma 5.11.2. For $A \in SL(n, C)$, $D \in \mathfrak{sl}(n, C)$ and $\mathbf{u}, \mathbf{v} \in \Lambda^k(C^n)$, we have

- (1) $A(\mathbf{u} \times \mathbf{v})A^{-1} = A\mathbf{u} \times {}^tA^{-1}\mathbf{v}, \quad [D, \mathbf{u} \times \mathbf{v}] = D\mathbf{u} \times \mathbf{v} + \mathbf{u} \times (-{}^tD\mathbf{v}).$
- (2) ${}^t(\mathbf{u} \times \mathbf{v}) = \mathbf{v} \times \mathbf{u}, \quad \tau(\mathbf{u} \times \mathbf{v}) = \tau(\mathbf{u}) \times \tau(\mathbf{v}).$
- (3) $\text{tr}(D(\mathbf{u} \times \mathbf{v})) = (-1)^{n-k} (D\mathbf{u}, \mathbf{v}).$

Now, we construct another C -Lie algebra \mathfrak{e}_8^C of type E_8 .

Theorem 5.11.3. In an $80 + 84 + 84 = 248$ dimensional C -vector space

$$\mathfrak{e}_8^C = \mathfrak{sl}(9, C) \oplus \Lambda^3(C^9) \oplus \Lambda^3(C^9),$$

we define a Lie bracket $[R_1, R_2]$ by

$$[(D_1, \mathbf{u}_1, \mathbf{v}_1), (D_2, \mathbf{u}_2, \mathbf{v}_2)] = (D, \mathbf{u}, \mathbf{v}),$$

where

$$\begin{cases} D = [D_1, D_2] + \mathbf{u}_1 \times \mathbf{v}_2 - \mathbf{u}_2 \times \mathbf{v}_1 \\ \mathbf{u} = D_1\mathbf{u}_2 - D_2\mathbf{u}_1 + *(\mathbf{v}_1 \wedge \mathbf{v}_2) \\ \mathbf{v} = -{}^tD_1\mathbf{v}_2 + {}^tD_2\mathbf{v}_1 - *(\mathbf{u}_1 \wedge \mathbf{u}_2), \end{cases}$$

then \mathfrak{e}_8^C becomes a C -Lie algebra of type E_8 .

Proof. In order to prove the Jacobi identity, we need the following Lemma.

Lemma 5.11.4. For $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \Lambda^3(C^9)$, we have

- (1) $\mathbf{u} \times *(\mathbf{v} \wedge \mathbf{w}) + \mathbf{v} \times *(\mathbf{w} \wedge \mathbf{u}) + \mathbf{w} \times *(\mathbf{u} \wedge \mathbf{v}) = 0,$
- (2) $(\mathbf{u} \times \mathbf{w})\mathbf{v} - (\mathbf{v} \times \mathbf{w})\mathbf{u} + *(\mathbf{u} \times \mathbf{v}) \wedge \mathbf{w} = 0.$

Proof. Let $\mathbf{u} = \mathbf{u}_1 \wedge \mathbf{u}_2 \wedge \mathbf{u}_3$, $\mathbf{v} = \mathbf{u}_4 \wedge \mathbf{u}_5 \wedge \mathbf{u}_6$ and $\mathbf{w} = \mathbf{u}_7 \wedge \mathbf{u}_8 \wedge \mathbf{u}_9$. For $\mathbf{x}, \mathbf{y} \in C^9$, we have

$$\begin{aligned} ((\mathbf{u} \times \mathbf{v})\mathbf{x}, \mathbf{y}) &= (*(\mathbf{v} \wedge *(\mathbf{u} \wedge \mathbf{x})), \mathbf{y}) + \frac{2}{3}(\mathbf{u}, \mathbf{v})(\mathbf{x}, \mathbf{y}) \\ &= -(\mathbf{x} \wedge \mathbf{u}, \mathbf{y} \wedge \mathbf{v}) + \frac{2}{3}(\mathbf{u}, \mathbf{v})(\mathbf{x}, \mathbf{y}) \\ &= (\mathbf{x} \wedge \mathbf{u}_2 \wedge \mathbf{u}_3, \mathbf{v})(\mathbf{u}_1, \mathbf{y}) - (\mathbf{x} \wedge \mathbf{u}_1 \wedge \mathbf{u}_3, \mathbf{v})(\mathbf{u}_2, \mathbf{y}) \\ &\quad + (\mathbf{x} \wedge \mathbf{u}_1 \wedge \mathbf{u}_2, \mathbf{v})(\mathbf{u}_3, \mathbf{y}) - \frac{1}{3}(\mathbf{u}, \mathbf{v})(\mathbf{x}, \mathbf{y}). \end{aligned}$$

Hence

$$\begin{aligned} (\mathbf{u} \times \mathbf{v})\mathbf{x} &= (\mathbf{x} \wedge \mathbf{u}_2 \wedge \mathbf{u}_3, \mathbf{v})\mathbf{u}_1 + (\mathbf{u}_1 \wedge \mathbf{x} \wedge \mathbf{u}_3, \mathbf{v})\mathbf{u}_2 \\ &\quad + (\mathbf{u}_1 \wedge \mathbf{u}_2 \wedge \mathbf{x}, \mathbf{v})\mathbf{u}_3 - \frac{1}{3}(\mathbf{u}, \mathbf{v})\mathbf{x}. \end{aligned} \tag{i}$$

Using this identity,

$$\begin{aligned} &(\mathbf{u} \times *(\mathbf{v} \wedge \mathbf{w}) + \mathbf{v} \times *(\mathbf{w} \wedge \mathbf{u}) + \mathbf{w} \times *(\mathbf{u} \wedge \mathbf{v}))\mathbf{x} \\ &= \sum_{j=1}^9 (\mathbf{u}_1 \wedge \cdots \wedge \mathbf{u}_{j-1} \wedge \mathbf{x} \wedge \mathbf{u}_{j+1} \wedge \cdots \wedge \mathbf{u}_9, \mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_9)\mathbf{u}_j \\ &\quad - (\mathbf{u}_1 \wedge \cdots \wedge \mathbf{u}_9, \mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_9)\mathbf{x} = \text{(ii)}. \end{aligned}$$

Denote $\mathbf{x} = \sum_{i=1}^9 x_i \mathbf{e}_i$, $\mathbf{u}_j = \sum_{k=1}^9 u_{jk} \mathbf{e}_k$ and $U = (u_{jk}) \in M(9, C)$. Hence we have

$$(\mathbf{u}_1 \wedge \cdots \wedge \mathbf{u}_{j-1} \wedge \mathbf{x} \wedge \mathbf{u}_{j+1} \wedge \cdots \wedge \mathbf{u}_9, \mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_9) = \sum_{k=1}^9 \tilde{u}_{jk} x_k,$$

$$(\mathbf{u}_1 \wedge \cdots \wedge \mathbf{u}_9, \mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_9) = \det U,$$

where \tilde{u}_{jk} is the cofactor of u_{jk} of the matrix U . Therefore

$$\begin{aligned} \text{(ii)} &= \sum_{j,k} x_k \tilde{u}_{jk} \mathbf{u}_j - (\det U)\mathbf{x} = \sum_{i,j,k} x_k \tilde{u}_{jk} u_{ji} \mathbf{e}_i - (\det U)\mathbf{x} \\ &= \sum_{j,k} x_k (\det U) \delta_{ki} \mathbf{e}_i - (\det U)\mathbf{x} = 0. \end{aligned}$$

Thus (1) is proved. Next, let $\mathbf{u} = \mathbf{u}_1 \wedge \mathbf{u}_2 \wedge \mathbf{u}_3$ and $\mathbf{v} = \mathbf{v}_1 \wedge \mathbf{v}_2 \wedge \mathbf{v}_3$. Using (i), for any $\mathbf{a} \in \Lambda^3(C^9)$, we have

$$\begin{aligned} &((\mathbf{u} \times \mathbf{w})\mathbf{v} - (\mathbf{v} \times \mathbf{w})\mathbf{u}, \mathbf{a}) \\ &= (((\mathbf{u} \times \mathbf{w})\mathbf{v}_1) \wedge \mathbf{v}_2 \wedge \mathbf{v}_3, \mathbf{a}) - (((\mathbf{u} \times \mathbf{w})\mathbf{v}_2) \wedge \mathbf{v}_1 \wedge \mathbf{v}_3, \mathbf{a}) \\ &\quad + (((\mathbf{u} \times \mathbf{w})\mathbf{v}_3) \wedge \mathbf{v}_1 \wedge \mathbf{v}_2, \mathbf{a}) - (((\mathbf{v} \times \mathbf{w})\mathbf{u}_1) \wedge \mathbf{u}_2 \wedge \mathbf{u}_3, \mathbf{a}) \\ &\quad + (((\mathbf{v} \times \mathbf{w})\mathbf{u}_2) \wedge \mathbf{u}_1 \wedge \mathbf{u}_3, \mathbf{a}) - (((\mathbf{v} \times \mathbf{w})\mathbf{u}_3) \wedge \mathbf{u}_1 \wedge \mathbf{u}_2, \mathbf{a})) \\ &= -(\mathbf{u}, \mathbf{w})(\mathbf{v}, \mathbf{a}) + \sum_{i=1}^3 \sum_{j=1}^3 (\mathbf{u}_i \wedge \mathbf{u}_{i+1} \wedge \mathbf{v}_j, \mathbf{w})(\mathbf{u}_{i+2} \wedge \mathbf{v}_{i+1} \wedge \mathbf{v}_{j+2}, \mathbf{a}) \end{aligned}$$

$$\begin{aligned}
& +(\mathbf{v}, \mathbf{w})(\mathbf{u}, \mathbf{a}) - \sum_{i=1}^3 \sum_{j=1}^3 (\mathbf{u}_i \wedge \mathbf{v}_j \wedge \mathbf{v}_{j+1}, \mathbf{w})(\mathbf{u}_{i+1} \wedge \mathbf{u}_{i+2} \wedge \mathbf{v}_{j+2}, \mathbf{a}) \\
& = -(\mathbf{u} \wedge \mathbf{v}, \mathbf{w} \wedge \mathbf{a}) = -(*(*(\mathbf{u} \wedge \mathbf{v}) \wedge \mathbf{w}), \mathbf{a}).
\end{aligned}$$

Thus (2) is proved.

From Lemmas 5.11.1, 5.11.2 and 5.11.4, we can prove that \mathfrak{e}_8^C is a C -Lie algebra. Therefore Theorem 5.11.3 is proved.

We will show that the Lie algebra \mathfrak{e}_8^C given in Theorem 5.11.3 is also the Lie algebra of type E_8 . Since we can not give explicit isomorphism between this \mathfrak{e}_8^C and \mathfrak{e}_8^C of Theorem 5.1.1, we shall show that this \mathfrak{e}_8^C is simple.

Theorem 5.11.5. $\mathfrak{e}_8^C = \mathfrak{sl}(9, C) \oplus \Lambda^3(C^9) \oplus \Lambda^3(C^9)$ is a simple C -Lie algebra of type E_8 .

Proof. We use the decomposition

$$\mathfrak{e}_8^C = \mathfrak{sl}(9, C) \oplus \mathfrak{q}, \quad \mathfrak{q} = \Lambda^3(C^9) \oplus \Lambda^3(C^9).$$

For a subset $I = \{i, j, k\}$ ($i < j < k$) of $\{1, 2, \dots, 9\}$, we put

$$\mathbf{e}_I = \mathbf{e}_i \wedge \mathbf{e}_j \wedge \mathbf{e}_k \in \Lambda^3(C^9).$$

Now, let \mathfrak{a} be a non-zero ideal of $\mathfrak{g} = \mathfrak{e}_8^C$.

(1) Case $\mathfrak{sl}(9, C) \cap \mathfrak{a} = \{0\}$ and $\mathfrak{q} \cap \mathfrak{a} = \{0\}$. Let $p : \mathfrak{g} \rightarrow \mathfrak{sl}(9, C)$ be the projection. If $p(\mathfrak{a}) = 0$, then \mathfrak{a} is contained in \mathfrak{q} , which contradicts $\mathfrak{q} \cap \mathfrak{a} = \{0\}$. Hence, $p(\mathfrak{a})$ is a non-zero ideal of $\mathfrak{sl}(9, C)$, so we have $p(\mathfrak{a}) = \mathfrak{sl}(9, C)$. For an element $D = \sum_{i=1}^8 H_i \in \mathfrak{sl}(9, C)$, $H_i = E_{ii} - E_{99}$, there exists an element $(\mathbf{u}, \mathbf{v}) = (\sum_I u_I \mathbf{e}_I, \sum_J v_J \mathbf{e}_J) \in \mathfrak{q}$ such that $(D, \mathbf{u}, \mathbf{v}) \in \mathfrak{a}$. Since $[(D, 0, 0), (X, \mathbf{u}, \mathbf{v})] = (0, D\mathbf{u}, -{}^t D\mathbf{v}) \in \mathfrak{q} \cap \mathfrak{a} = \{0\}$, we have

$$\begin{aligned}
0 &= D\mathbf{u} = \sum_I u_I D\mathbf{e}_I = 3 \sum_{I \not\ni 9} u_I \mathbf{e}_I - 6 \sum_{I \ni 9} u_I \mathbf{e}_I, \\
0 &= -{}^t D\mathbf{v} = -3 \sum_{J \not\ni 9} v_J \mathbf{e}_J + 6 \sum_{J \ni 9} v_J \mathbf{e}_J,
\end{aligned}$$

i.e., $u_I = 0$ and $v_J = 0$. Then, $0 \neq (D, \mathbf{u}, \mathbf{v}) = (D, 0, 0) \in \mathfrak{sl}(9, C) \cap \mathfrak{a} = \{0\}$. This is a contradiction.

(2) Case $\mathfrak{sl}(9, C) \cap \mathfrak{a} \neq \{0\}$. Since $\mathfrak{sl}(9, C) \cap \mathfrak{a}$ is a non-zero ideal of $\mathfrak{sl}(9, C)$, we have $\mathfrak{sl}(9, C) \subset \mathfrak{a}$. For any $\mathbf{e}_i \wedge \mathbf{e}_j \wedge \mathbf{e}_k \in \Lambda^3(C^9)$, put

$$D = \frac{1}{3}(E_{ii} + E_{jj} + E_{kk}) - E_{ll}, \quad l \neq i, j, k.$$

Since $(D, 0, 0) \in \mathfrak{sl}(9, C) \subset \mathfrak{a}$, we see that

$$\begin{aligned}
(0, \mathbf{e}_i \wedge \mathbf{e}_j \wedge \mathbf{e}_k, 0) &= [(D, 0, 0), (0, \mathbf{e}_i \wedge \mathbf{e}_j \wedge \mathbf{e}_k, 0)] \in \mathfrak{a}, \\
(0, 0, \mathbf{e}_i \wedge \mathbf{e}_j \wedge \mathbf{e}_k) &= [(D, 0, 0), (0, 0, -\mathbf{e}_i \wedge \mathbf{e}_j \wedge \mathbf{e}_k)] \in \mathfrak{a}.
\end{aligned}$$

It follows that $\mathfrak{q} \subset \mathfrak{a}$. Hence we have $\mathfrak{a} = \mathfrak{g}$.

(3) Case $\mathfrak{q} \cap \mathfrak{a} \neq \{0\}$. Let $R = (0, \mathbf{u}, \mathbf{v})$ be a non-zero element of $\mathfrak{q} \cap \mathfrak{a}$. In the case $\mathbf{u} \neq 0$, we put $\mathbf{u} = \sum_I u_I \mathbf{e}_I$. Without loss of generality, we may assume that $u_{\{123\}} = 1$. Putting $S_{ij} = (E_{ii} - E_{jj}, 0, 0) \in \mathfrak{g}$ and $T = (0, 0, \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_4) \in \mathfrak{g}$, we have

$$\begin{aligned} 0 &\neq \text{ad}(T)\text{ad}(S_{37})\text{ad}(S_{27})\text{ad}(S_{17})\text{ad}(S_{36})\text{ad}(S_{25})\text{ad}(S_{14})R \\ &= (-E_{34}, 0, 0) \in \mathfrak{sl}(9, C) \cap \mathfrak{a}. \end{aligned}$$

Then we can reduce this case to the case (2). In case $\mathbf{v} \neq 0$, we can similarly reduce to the case (2).

Thus the simplicity of \mathfrak{g} has been proved. Since the dimension of \mathfrak{g} is 248, we see that \mathfrak{g} is a Lie algebra of type E_8 .

Proposition 5.11.6. *The Killing form B_8 of the Lie algebra $\mathfrak{e}_8^C = \mathfrak{sl}(9, C) \oplus \Lambda^3(C^9) \oplus \Lambda^3(C^9)$ is given by*

$$B_8((D_1, \mathbf{u}_1, \mathbf{v}_1), (D_2, \mathbf{u}_2, \mathbf{v}_2)) = 60(\text{tr}(D_1 D_2) + (\mathbf{u}_1, \mathbf{v}_2) + (\mathbf{u}_2, \mathbf{v}_1)).$$

Proof. We consider a symmetric bilinear form B of \mathfrak{e}_8^C :

$$B((D_1, \mathbf{u}_1, \mathbf{v}_1), (D_2, \mathbf{u}_2, \mathbf{v}_2)) = \text{tr}(D_1 D_2) + (\mathbf{u}_1, \mathbf{v}_2) + (\mathbf{u}_2, \mathbf{v}_1).$$

Using Lemmas 5.11.2, 5.11.4, we see that B is \mathfrak{e}_8^C -adjoint invariant. Since \mathfrak{e}_8^C is simple, there exists $k \in C$ such that $B_8(R_1, R_2) = kB(R_1, R_2)$ for all $R_i \in \mathfrak{e}_8^C$. To determine k , let $R = R_1 = R_2 = (E_{11} - E_{22}, 0, 0) \in \mathfrak{e}_8^C$. Then we have

$$B_8(R, R) = 120, \quad B(R, R) = 2.$$

Therefore $k = 60$.

We define a complex-conjugate linear transformation $\tau\tilde{\lambda}$ of \mathfrak{e}_8^C by

$$\tau\tilde{\lambda}(D, \mathbf{u}, \mathbf{v}) = (-\tau^t D, -\tau\mathbf{v}, -\tau\mathbf{u}).$$

and we define a Hermitian inner product $\langle R_1, R_2 \rangle$ in \mathfrak{e}_8^C by

$$\langle R_1, R_2 \rangle = -B_8(R_1, \tau\tilde{\lambda}R_2).$$

Then we have

$$\langle R_1, R_2 \rangle = 60(\text{tr}(D_1 \tau^t D_2) + (\mathbf{u}_1, \tau\mathbf{u}_2) + (\mathbf{v}_2, \tau\mathbf{v}_1)).$$

As in Theorem 5.5.3,

$$E_8 = \{\alpha \in \text{Aut}(\mathfrak{e}_8^C) \mid \langle \alpha R_1, \alpha R_2 \rangle = \langle R_1, R_2 \rangle\}$$

is a simply connected compact simple Lie group of type E_8 .

We define a C -linear transformation w_3 of \mathfrak{e}_8^C by

$$w_3(D, \mathbf{u}, \mathbf{v}) = (D, \omega \mathbf{u}, \omega^2 \mathbf{v}),$$

where $\omega = -\frac{1}{2} + \frac{\sqrt{3}}{2}i \in C$. Then, $w_3 \in E_8$ and $w_3^3 = 1$.

Now, we study the following subgroup $(E_8)^{w_3}$ of E_8 :

$$(E_8)^{w_3} = \{\alpha \in E_8 \mid w_3 \alpha = \alpha w_3\}.$$

Theorem 5.11.7. $(E_8)^{w_3} \cong SU(9)/\mathbf{Z}_3$, $\mathbf{Z}_3 = \{E, \omega E, \omega^2 E\}$.

Proof. We define a mapping $\varphi : SU(9) \rightarrow (E_8)^{w_3}$ by

$$\varphi(A)(D, \mathbf{u}, \mathbf{v}) = (ADA^{-1}, A\mathbf{u}, {}^t A^{-1} \mathbf{v}).$$

φ is well-defined: $\varphi(A) \in (E_8)^{w_3}$. Indeed, for $A = \exp X, X \in \mathfrak{su}(9)$, we have

$$\begin{aligned} \exp(\text{ad}(X, 0, 0))(D, \mathbf{u}, \mathbf{v}) &= (\exp(\text{ad}(X)D), (\exp X)\mathbf{u}, (\exp(-{}^t X))\mathbf{v}) \\ &= (\text{Ad}(\exp X)D, (\exp X)\mathbf{u}, {}^t(\exp X)^{-1} \mathbf{v}) \\ &= \varphi(\exp X)(D, \mathbf{u}, \mathbf{v}). \end{aligned}$$

Hence $\varphi(A) \in E_8$. Clearly $w_3 \varphi(A) = \varphi(A) w_3$. Therefore $\varphi(A) \in (E_8)^{w_3}$. Obviously φ is a homomorphism. We shall show that φ is onto. Since the Lie algebra $(\mathfrak{e}_8)^{w_3}$ of the group $(E_8)^{w_3}$ is

$$(\mathfrak{e}_8)^{w_3} = \{R \in \mathfrak{e}_8^C \mid \tau \tilde{\lambda} R = R, w_3 R = R\} = \{(D, 0, 0) \in \mathfrak{e}_8^C \mid D \in \mathfrak{su}(9)\} \cong \mathfrak{su}(9),$$

the differential φ_* of φ is onto. Since $(E_8)^{w_3}$ is connected, φ is also onto. It is not difficult to see that $\ker \varphi = \{E, \omega E, \omega^2 E\} = \mathbf{Z}_3$. Thus we have the isomorphism $SU(9)/\mathbf{Z}_3 \cong (E_8)^{w_3}$.

5.12. Automorphism z_5 of order 5 and subgroup $(SU(5) \times SU(5))/\mathbf{Z}_5$ of E_8

We shall construct one more C -Lie algebra of type E_8 .

Theorem 5.12.1. *In a $48 + 50 \times 4 = 248$ dimensional C -vector space*

$$\mathfrak{e}_8^C = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1},$$

(suffices are considered mod 5) *where*

$$\begin{aligned} \mathfrak{g}_0 &= \mathfrak{sl}(5, C) \oplus \mathfrak{sl}(5, C), \\ \mathfrak{g}_1 &= C^5 \otimes \Lambda^2(C^5) = \mathfrak{g}_{-1}, \quad \mathfrak{g}_2 = \Lambda^2(C^5) \otimes C^5 = \mathfrak{g}_{-2}, \end{aligned}$$

we define a Lie bracket $[R_1, R_2]$ as follows.

$$\begin{aligned}
[\mathfrak{g}_0, \mathfrak{g}_0] &\subset \mathfrak{g}_0 & [(C_1, D_1), (C_2, D_2)] &= ([C_1, C_2], [D_1, D_2]), \\
[\mathfrak{g}_0, \mathfrak{g}_1] &\subset \mathfrak{g}_1 & [(C, D), x \otimes a] &= (Cx) \otimes a + x \otimes (Da), \\
[\mathfrak{g}_0, \mathfrak{g}_2] &\subset \mathfrak{g}_2 & [(C, D), b \otimes y] &= (Cb) \otimes y + b \otimes (-{}^tDa), \\
[\mathfrak{g}_0, \mathfrak{g}_{-2}] &\subset \mathfrak{g}_{-2} & [(C, D), c \otimes z] &= (-{}^tCc) \otimes z + c \otimes (Dz), \\
[\mathfrak{g}_0, \mathfrak{g}_{-1}] &\subset \mathfrak{g}_{-1} & [(C, D), w \otimes d] &= (-{}^tCw) \otimes z + c \otimes (-{}^tDz), \\
[\mathfrak{g}_1, \mathfrak{g}_{-1}] &\subset \mathfrak{g}_0 & [x \otimes a, w \otimes d] &= -(a, d)x \times w, (x, w)a \times d, \\
[\mathfrak{g}_2, \mathfrak{g}_{-2}] &\subset \mathfrak{g}_0 & [b \otimes y, c \otimes z] &= ((y, z)b \times c, (b, c)z \times y), \\
[\mathfrak{g}_1, \mathfrak{g}_1] &\subset \mathfrak{g}_2 & [x_1 \otimes a_1, x_2 \otimes a_2] &= (x_1 \wedge x_2) \otimes *(a_1 \wedge a_2), \\
[\mathfrak{g}_{-1}, \mathfrak{g}_{-1}] &\subset \mathfrak{g}_{-2} & [b_1 \otimes y_1, b_2 \otimes y_2] &= *(b_1 \wedge b_2) \otimes (y_1 \wedge y_2), \\
[\mathfrak{g}_2, \mathfrak{g}_2] &\subset \mathfrak{g}_{-1} & [x \otimes a, b \otimes y] &= *(b \wedge x) \otimes *(a \wedge y), \\
[\mathfrak{g}_{-2}, \mathfrak{g}_{-2}] &\subset \mathfrak{g}_1 & [b \otimes y, w \otimes d] &= *(b \wedge w) \otimes *(d \wedge y). \\
[\mathfrak{g}_1, \mathfrak{g}_2] &\subset \mathfrak{g}_{-2} & & \\
[\mathfrak{g}_{-1}, \mathfrak{g}_{-2}] &\subset \mathfrak{g}_2 & & \\
[\mathfrak{g}_2, \mathfrak{g}_{-1}] &\subset \mathfrak{g}_1 & & \\
[\mathfrak{g}_{-2}, \mathfrak{g}_1] &\subset \mathfrak{g}_{-1} & &
\end{aligned}$$

Then \mathfrak{e}_8^C becomes a C -Lie algebra.

Proof. In order to prove the Jacobi identity, we need the following Lemma.

Lemma 5.12.2. For $x, y, z \in \Lambda^1(C^5) = C^5$ and $a, b, c \in \Lambda^2(C^5)$, we have

- (1) $*a \wedge *(b \wedge c) + *b \wedge *(c \wedge a) + *c \wedge *(a \wedge b) = 0$,
- (2) $*(a \wedge *(b \wedge x)) + *(b \wedge *(a \wedge x)) + x \wedge *(a \wedge b) = 0$,
- (3) $*(x \wedge y) \wedge z = (x, z)y - (y, z)x$,
- (4) $x \wedge *(a \wedge y) + *(y \wedge (a \wedge x)) - (x, y)a = 0$,
- (5) $*(a \wedge *(b \wedge x)) - *(b \wedge *(a \wedge x)) - (a, b)x = 0$,
- (6) $a \times *(b \wedge x) + b \times *(a \wedge x) - x \times *(a \wedge b) = 0$,
- (7) $*(a \wedge x) \times y - *(a \wedge y) \times x + a \times (x \wedge y) = 0$,
- (8) $(a \wedge b)c = *(a \wedge c) \wedge b - \frac{1}{5}(a, b)c - (b, c)a$,
- (9) $(x \times y)a = -*(y \wedge *(x \wedge a)) + \frac{3}{5}(x, y)a$.

Proof. (1) Let $a = a_1 \wedge a_2, b = a_3 \wedge a_4, c = a_5 \wedge a_6$ and $a_i = \sum_{j=1}^5 a_{ij}e_j$. Since

$$\begin{aligned}
&(*a \wedge *(b \wedge c), x) = (a, *(b \wedge c) \wedge x) \\
&= (a_1, *(b \wedge c))(a_2, x) - (a_2, *(b \wedge c))(a_1, x) \\
&= (a_1 \wedge b \wedge c, e_1 \wedge \cdots \wedge e_5)(a_2, x) - (a_2 \wedge b \wedge c, e_1 \wedge \cdots \wedge e_5)(a_1, x),
\end{aligned}$$

we have

$$*(a \wedge *(b \wedge c) + *b \wedge *(c \wedge a) + *c \wedge *(a \wedge b))$$

$$\begin{aligned}
&= \sum_{j=1}^5 \sum_{i=1}^6 (-1)^i (\mathbf{a}_1 \wedge \cdots \wedge \mathbf{a}_{i-1} \wedge \mathbf{a}_{i+1} \wedge \cdots \wedge \mathbf{a}_6, \mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_5) a_{ij} \mathbf{e}_j \\
&= - \sum_{j=1}^5 \det \begin{pmatrix} a_{1j} & a_{11} & \cdots & a_{15} \\ a_{2j} & a_{21} & \cdots & a_{25} \\ & & \cdots & \\ a_{6j} & a_{61} & \cdots & a_{65} \end{pmatrix} \mathbf{e}_j = 0.
\end{aligned}$$

(2) Using (1), we have

$$\begin{aligned}
&(*(\mathbf{a} \wedge *(\mathbf{b} \wedge \mathbf{x})) + *(\mathbf{b} \wedge *(\mathbf{a} \wedge \mathbf{x})) + \mathbf{x} \wedge *(\mathbf{a} \wedge \mathbf{b}), \mathbf{c}) \\
&= (*\mathbf{b} \wedge \mathbf{x}, \mathbf{c} \wedge \mathbf{a}) + (*\mathbf{a} \wedge \mathbf{x}, \mathbf{b} \wedge \mathbf{c}) + (*\mathbf{c} \wedge \mathbf{x}, \mathbf{a} \wedge \mathbf{b}) \\
&= (*\mathbf{x}, *\mathbf{b} \wedge *(\mathbf{c} \wedge \mathbf{a}) + *\mathbf{a} \wedge *(\mathbf{b} \wedge \mathbf{c}) + *\mathbf{c} \wedge *(\mathbf{a} \wedge \mathbf{b})) = 0.
\end{aligned}$$

(3) For any $\mathbf{v} \in \Lambda^1(C^5) = C^5$, we have

$$(*(\mathbf{x} \wedge \mathbf{y}) \wedge \mathbf{z}, \mathbf{v}) = (\mathbf{x} \wedge \mathbf{y}, \mathbf{z} \wedge \mathbf{v}) = (\mathbf{x}, \mathbf{z})(\mathbf{y}, \mathbf{v}) - (\mathbf{y}, \mathbf{z})(\mathbf{x}, \mathbf{v}).$$

This shows (3).

(4),(5) Let $\mathbf{a} = \mathbf{a}_1 \wedge \mathbf{a}_2$. Since

$$\begin{aligned}
(\mathbf{x} \wedge \mathbf{a}, \mathbf{y} \wedge \mathbf{b}) &= (\mathbf{x}, \mathbf{y})(\mathbf{a}, \mathbf{b}) - (\mathbf{a}_1, \mathbf{y})(\mathbf{x} \wedge \mathbf{a}_2, \mathbf{b}) + (\mathbf{a}_2, \mathbf{y})(\mathbf{x} \wedge \mathbf{a}_1, \mathbf{b}), \\
(*\mathbf{b} \wedge \mathbf{x}, *\mathbf{a} \wedge \mathbf{y}) &= (\mathbf{a}, \mathbf{y} \wedge *(\mathbf{b} \wedge \mathbf{x})) = (\mathbf{a}_1, \mathbf{y})(\mathbf{x} \wedge \mathbf{a}_2, \mathbf{b}) - (\mathbf{a}_2, \mathbf{y})(\mathbf{x} \wedge \mathbf{a}_1, \mathbf{b}),
\end{aligned}$$

we have

$$(\mathbf{x} \wedge \mathbf{a}, \mathbf{y} \wedge \mathbf{b}) + (*\mathbf{b} \wedge \mathbf{x}, *\mathbf{a} \wedge \mathbf{y}) = (\mathbf{x}, \mathbf{y})(\mathbf{a}, \mathbf{b}).$$

Using this identity, we have

$$\begin{aligned}
&(\mathbf{x} \wedge *(\mathbf{a} \wedge \mathbf{y}) + *(\mathbf{y} \wedge *(\mathbf{a} \wedge \mathbf{x})) - (\mathbf{x}, \mathbf{y})\mathbf{a}, \mathbf{b}) \\
&= (*\mathbf{b} \wedge \mathbf{x}, *\mathbf{a} \wedge \mathbf{y}) + (\mathbf{x} \wedge \mathbf{a}, \mathbf{y} \wedge \mathbf{b}) - (\mathbf{x}, \mathbf{y})(\mathbf{a}, \mathbf{b}) = 0, \\
&(*(\mathbf{a} \wedge *(\mathbf{b} \wedge \mathbf{x})) - *(\mathbf{b} \wedge *(\mathbf{a} \wedge \mathbf{x})) - (\mathbf{a}, \mathbf{b})\mathbf{x}, \mathbf{y}) \\
&= (\mathbf{x} \wedge \mathbf{b}, \mathbf{y} \wedge \mathbf{a}) + (*\mathbf{a} \wedge \mathbf{x}, *\mathbf{b} \wedge \mathbf{y}) - (\mathbf{x}, \mathbf{y})(\mathbf{a}, \mathbf{b}) = 0.
\end{aligned}$$

(6) Since

$$((\mathbf{x} \times \mathbf{y})\mathbf{z}, \mathbf{v}) = -(\mathbf{x} \wedge \mathbf{z}, \mathbf{y} \wedge \mathbf{v}) + \frac{4}{5}(\mathbf{x}, \mathbf{y})(\mathbf{z}, \mathbf{v}) = (\mathbf{y}, \mathbf{z})(\mathbf{x}, \mathbf{v}) - \frac{1}{5}(\mathbf{x}, \mathbf{y})(\mathbf{z}, \mathbf{v}),$$

we have

$$(\mathbf{x} \times \mathbf{y})\mathbf{z} = (\mathbf{y}, \mathbf{z})\mathbf{x} - \frac{1}{5}(\mathbf{x}, \mathbf{y})\mathbf{z}. \quad (\text{i})$$

For $\mathbf{v}, \mathbf{w} \in \Lambda^1(C^5) = C^5$, we have

$$\begin{aligned}
&((\mathbf{a} \times *(\mathbf{b} \wedge \mathbf{x}))\mathbf{v}, \mathbf{w}) \\
&= (\mathbf{v} \wedge \mathbf{a}, \mathbf{w} \wedge *(\mathbf{b} \wedge \mathbf{x})) - \frac{3}{5}(*(\mathbf{a} \wedge \mathbf{b}), \mathbf{x})(\mathbf{v}, \mathbf{w})
\end{aligned}$$

$$\begin{aligned}
&= (v, w)(a, *(b \wedge x)) - (a_1, w)(w \wedge a_2, *(b \wedge x)) \\
&\quad + (a_2, w)(w \wedge a_1, *(b \wedge x)) - \frac{3}{5}(*(a \wedge b), x)(v, w) \\
&= \frac{2}{5}(*(a \wedge b), x)(v, w) - (a_1, w)(b \wedge x \wedge w \wedge a_2, e_1 \wedge \cdots \wedge e_5) \\
&\quad + (a_2, w)(b \wedge x \wedge w \wedge a_1, e_1 \wedge \cdots \wedge e_5), \\
&((b \times *(a \wedge x))v, w) \\
&= (x \wedge a, w \wedge *(b \wedge v)) - \frac{3}{5}(*(a \wedge b), x)(v, w) \\
&= (v, w)(*(a \wedge b), v) + (a_1, w)(b \wedge x \wedge w \wedge a_2, e_1 \wedge \cdots \wedge e_5) \\
&\quad - (a_2, w)(b \wedge x \wedge w \wedge a_1, e_1 \wedge \cdots \wedge e_5) - \frac{3}{5}(*(a \wedge b), x)(v, w).
\end{aligned}$$

Using (i), we have

$$\begin{aligned}
(a \times *(b \wedge x))v + (b \times *(a \wedge x))v &= (x, w) * (a \wedge b) - \frac{1}{5}(*(a \wedge b), x)v \\
&= (x \times *(a \wedge b))v.
\end{aligned}$$

(7) We have

$$\begin{aligned}
((a \times (x \wedge y))v, w) &= (x \wedge y \wedge w, v \wedge a) - \frac{3}{5}(a, x \wedge y)(v, w) \\
&= (x, v)(y \wedge w, a) - (y, v)(x \wedge w, a) + \frac{2}{5}(a, x \wedge y)(v, w) \\
&= (x, v)(*(a \wedge y), w) - (y, v)(*(a \wedge x), w) + \frac{2}{5}(a, x \wedge y)(v, w).
\end{aligned}$$

On the other hand, using (i), we have

$$\begin{aligned}
(*(a \wedge x) \times y)v &= (y, v) * (*a \wedge x) - \frac{1}{5}(*(a \wedge x), v)y, \\
&= (y, v) * (*a \wedge x) - \frac{1}{5}(a, x \wedge v)y, \\
-(*(a \wedge x) \times y)v &= -(x, v) * (*a \wedge y) - \frac{1}{5}(a, x \wedge v)y.
\end{aligned}$$

Hence (7) is proved.

(8) Let $a = a_1 \wedge a_2$ and $c = c_1 \wedge c_2$. Since

$$\begin{aligned}
((a \times b)v, w) &= (a \wedge v, b \wedge w) - \frac{3}{5}(a, b)(v, w) \\
&= -(a_1, w)(x \wedge a_2, b) + (a_2, w)(x \wedge a_1, b) + \frac{2}{5}(a, b)(v, w),
\end{aligned}$$

we have

$$\begin{aligned}
(a \times b)c &= -(a_1 \wedge c_1, b)a_1 \wedge c_2 + (a_1 \wedge c_2, b)a_2 \wedge c_1 \\
&\quad + (a_2 \wedge c_1, b)a_1 \wedge c_2 - (a_2 \wedge c_2, b)a_1 \wedge c_1 + \frac{4}{5}(a, b)c.
\end{aligned}$$

On the other hand, for $\mathbf{d} \in \Lambda^2(C^5)$, we have

$$\begin{aligned} (*(*(\mathbf{a} \wedge \mathbf{c}) \wedge \mathbf{b}), \mathbf{d}) &= (\mathbf{a} \wedge \mathbf{c}, \mathbf{b} \wedge \mathbf{d}) \\ &= (\mathbf{a}, \mathbf{b})(\mathbf{c}, \mathbf{d}) - (\mathbf{a}_1 \wedge \mathbf{c}_1, \mathbf{b})(\mathbf{a}_2 \wedge \mathbf{c}_2, \mathbf{d}) + (\mathbf{a}_1 \wedge \mathbf{c}_2, \mathbf{b})(\mathbf{a}_2 \wedge \mathbf{c}_1, \mathbf{d}) \\ &\quad + (\mathbf{a}_2 \wedge \mathbf{c}_1, \mathbf{b})(\mathbf{a}_1 \wedge \mathbf{c}_2, \mathbf{d}) - (\mathbf{a}_2 \wedge \mathbf{c}_2, \mathbf{b})(\mathbf{a}_1 \wedge \mathbf{c}_1, \mathbf{d}) + (\mathbf{c}, \mathbf{b})(\mathbf{a}, \mathbf{d}). \end{aligned}$$

Hence (8) is proved.

(9) Using (i), we have

$$(\mathbf{x} \times \mathbf{y})\mathbf{a} = (\mathbf{y}, \mathbf{a}_1)\mathbf{x} \wedge \mathbf{a}_2 - (\mathbf{y}, \mathbf{a}_2)\mathbf{x} \wedge \mathbf{a}_1 - \frac{2}{5}(\mathbf{x}, \mathbf{y})\mathbf{a}.$$

On the other hand, we have

$$\begin{aligned} (- * (\mathbf{y} \wedge *(\mathbf{x} \wedge \mathbf{a})), \mathbf{b}) &= -(\mathbf{x} \wedge \mathbf{a}, \mathbf{y} \wedge \mathbf{b}) \\ &= (\mathbf{y}, \mathbf{a}_1)(\mathbf{x} \wedge \mathbf{a}_2, \mathbf{b}) - (\mathbf{y}, \mathbf{a}_2)(\mathbf{x} \wedge \mathbf{a}_1, \mathbf{b}) - (\mathbf{x}, \mathbf{y})(\mathbf{a}, \mathbf{b}). \end{aligned}$$

Hence (9) is proved.

From Lemmas 5.11.1, 5.11.2 and 5.12.2, we can prove that \mathfrak{e}_8^C be comes a C -Lie algebra. Furthermore we have the following theorem.

Theorem 5.12.3. *The Lie algebra $\mathfrak{e}_8^C = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$ is a C -simple Lie algebra of type E_8 .*

Proof. We shall show that \mathfrak{e}_8^C is simple. For this end, we use the decomposition $\mathfrak{g} = \mathfrak{e}_8^C = \mathfrak{g}_{01} \oplus \mathfrak{g}_{12} \oplus \mathfrak{q}$, where

$$\begin{aligned} \mathfrak{g}_{01} &= \{(C, 0) \in \mathfrak{g}_0 \mid C \in \mathfrak{sl}(5, C)\} \cong \mathfrak{sl}(5, C), \\ \mathfrak{g}_{02} &= \{(0, D) \in \mathfrak{g}_0 \mid D \in \mathfrak{sl}(5, C)\} \cong \mathfrak{sl}(5, C), \\ \mathfrak{q} &= \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}. \end{aligned}$$

Now, let \mathfrak{a} be a non-zero ideal of \mathfrak{g} . There are three cases to be considered.

(1) Case $\mathfrak{g}_{01} \cap \mathfrak{a} = \{0\}$, $\mathfrak{g}_{02} \cap \mathfrak{a} = \{0\}$ and $\mathfrak{q} \cap \mathfrak{a} = \{0\}$. Let $p_i : \mathfrak{g} \rightarrow \mathfrak{g}_{0i}$ ($i = 1, 2$) denote the projection. If $p_1(\mathfrak{a}) = \{0\}$ and $p_2(\mathfrak{a}) = \{0\}$, then \mathfrak{a} is contained in \mathfrak{q} , which contradicts $\mathfrak{q} \cap \mathfrak{a} = \{0\}$. Hence, without loss of generality, we may assume that $p_1(\mathfrak{a}) = \mathfrak{g}_{01}$, because \mathfrak{g}_{01} is a simple Lie algebra. For $C = \sum_{i=1}^4 H_i \in \mathfrak{sl}(3, C)$ where $H_i = E_{ii} - E_{55}$, there exists $(D, g_1, g_2, g_{-2}, g_{-1}) \in \mathfrak{g}_{01} \oplus \mathfrak{q}$ such that $(C, D, g_1, g_2, g_{-2}, g_{-1}) \in \mathfrak{a}$. Since

$$\begin{aligned} [(C, 0), (C, D, g_1, g_2, g_{-2}, g_{-1})] \\ = (0, 0, [C, g_1], [C, g_2], [C, g_{-2}], [C, g_{-1}]) \in \mathfrak{q} \cap \mathfrak{a} = \{0\}, \end{aligned}$$

we have $[C, g_i] = 0$ ($i = 1, 2, -2, -1$). Since any eigenvalue of $\text{ad}X$ is not 0, we have $g_i = 0$. Then we have $(C, D) \in \mathfrak{g}_0 \cap \mathfrak{a}$. Since

$$[(C, D), (E_{45}, 0)] = (5E_{45}, 0) \in \mathfrak{g}_{01} \cap \mathfrak{a},$$

we have $\mathfrak{g}_{01} \cap \mathfrak{a} \neq \{0\}$. This is a contradiction.

(2) Case $\mathfrak{g}_{01} \cap \mathfrak{a} \neq \{0\}$ or $\mathfrak{g}_{02} \cap \mathfrak{a} \neq \{0\}$. We may assume that $\mathfrak{g}_{01} \cap \mathfrak{a} \neq \{0\}$. Since \mathfrak{g}_{01} is simple, we have $\mathfrak{g}_{01} \subset \mathfrak{a}$. Since $[\mathfrak{g}_{01}, \mathfrak{g}_i] = \mathfrak{g}_i$ ($i = 1, 2, -2, -1$), we have $\mathfrak{q} \subset \mathfrak{a}$. Since

$$\mathfrak{a} \supset [\mathfrak{g}_1, \mathfrak{g}_{-1}] \ni [e_1 \otimes (e_1 \wedge e_2), e_1 \otimes (e_1 \wedge e_3)] = (0, -E_{23}),$$

we have $\mathfrak{g}_{02} \cap \mathfrak{a} \neq \{0\}$. It follows that $\mathfrak{g}_{02} \subset \mathfrak{a}$. Hence we have $\mathfrak{a} = \mathfrak{g}$.

(3) Case $\mathfrak{q} \cap \mathfrak{a} \neq \{0\}$. Let $R = (g_1, g_2, g_{-2}, g_{-1})$ ($g_i \in \mathfrak{g}_i$) be a non-zero element of $\mathfrak{q} \cap \mathfrak{a}$. In the case $g_1 \neq 0$, we put $g_1 = \sum_{i,j < k} g_{ijk} e_i \otimes (e_j \wedge e_k)$. Without loss of generality, we may assume that $g_{112} \neq 0$. Putting $S_{ijkl} = (E_{ii} - E_{jj}, E_{kk} - E_{ll}) \in \mathfrak{g}_0$ and $T = e_2 \otimes e_1 \wedge e_2 \in \mathfrak{g}_{-1}$, we have

$$\text{ad}(T)\text{ad}(S_{1523})\text{ad}(S_{1415})\text{ad}(S_{1314})\text{ad}(S_{1213})R = (-E_{12}, 0) \in \mathfrak{g}_{01} \cap \mathfrak{a}.$$

Then we can reduce this case to the case (2). In the case $g_i \neq 0$ ($i = 2, -2, -1$), we can similarly reduce to the case (2).

Thus the simplicity of \mathfrak{g} has been proved. Since the dimension of \mathfrak{g} is 248, we see that \mathfrak{g} is a C -Lie algebra of type of E_8 .

Proposition 5.12.4. *The Killing form B_8 of the Lie algebra $\mathfrak{e}_8^C = \mathfrak{sl}(5, C) \oplus \mathfrak{sl}(5, C) \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$ is given by*

$$\begin{aligned} B_8(R_1, R_2) = & 60(\text{tr}(C_1 C_2) + \text{tr}(D_1 D_2) - (x_1, w_2)(a_1, d_2) - (x_2, w_1)(a_2, d_1) \\ & - (y_1, z_2)(b_1, c_2) - (y_2, z_1)(b_2, c_1)), \end{aligned}$$

where $R_i = (C_i, D_i, x_i \otimes a_i, b_i \otimes y_i, c_i \otimes z_i, w_i \otimes d_i) \in \mathfrak{e}_8^C$.

Proof. We consider a symmetric bilinear form B of \mathfrak{e}_8^C :

$$\begin{aligned} B(R_1, R_2) = & \text{tr}(C_1 C_2) + \text{tr}(D_1 D_2) - (x_1, w_2)(a_1, d_2) - (x_2, w_1)(a_2, d_1) \\ & - (y_1, z_2)(b_1, c_2) - (y_2, z_1)(b_2, c_1). \end{aligned}$$

Using Lemmas 5.12.1, 5.12.2, we see that B is \mathfrak{e}_8^C -adjoint invariant. Since \mathfrak{e}_8^C is simple, there exists $k \in C$ such that $B_8(R_1, R_2) = kB(R_1, R_2)$ for all $R_i \in \mathfrak{e}_8^C$. To determine k , let $R = R_1 = R_2 = (E_{11} - E_{22}, 0, 0, 0, 0, 0) \in \mathfrak{e}_8^C$. Then we have

$$B_8(R, R) = 120, \quad B(R, R) = 2.$$

Therefore $k = 60$.

We define a complex-conjugate linear transformation $\tau\tilde{\lambda}$ of \mathfrak{e}_8^C by

$$\begin{aligned} \tau\tilde{\lambda}(C, D, x \otimes a, b \otimes y, c \otimes z, w \otimes d) \\ = (-\tau^t C, -\tau^t D, \tau w \otimes \tau d, \tau c \otimes \tau z, \tau b \otimes \tau y, \tau x \otimes \tau a). \end{aligned}$$

Further we define a positive definite Hermitian inner product $\langle R_1, R_2 \rangle$ in \mathfrak{e}_8^C by

$$\langle R_1, R_2 \rangle = -B_8(R_1, \tau \tilde{\lambda} R_2).$$

Then, we have

$$\begin{aligned} \langle R_1, R_2 \rangle = & 60(\text{tr}(C_1 \tau^t C_2) + \text{tr}(D_1 \tau^t D_2) + (\mathbf{x}_1, \tau \mathbf{x}_2)(\mathbf{a}_1, \tau \mathbf{a}_2) + (\mathbf{y}_1, \tau \mathbf{y}_2)(\mathbf{b}_1, \tau \mathbf{b}_2) \\ & + (\mathbf{z}_1, \tau \mathbf{z}_2)(\mathbf{c}_1, \tau \mathbf{c}_2) + (\mathbf{w}_1, \tau \mathbf{w}_2)(\mathbf{d}_1, \tau \mathbf{d}_2)). \end{aligned}$$

As in Theorem 5.5.3,

$$E_8 = \{\alpha \in \text{Aut}(\mathfrak{e}_8^C) \mid \langle \alpha R_1, \alpha R_2 \rangle = \langle R_1, R_2 \rangle\}$$

is a simply connected compact simple Lie group of type E_8 .

Let $\zeta = \exp(2\pi i/5) \in C$ and we define a C -linear transformation z_5 of \mathfrak{e}_8^C by

$$z_5(C, D, g_1, g_2, g_{-2}, g_{-1}) = (C, D, \zeta(g_1), \zeta^2(g_2), \zeta^3(g_{-2}), \zeta^4(g_{-1})).$$

Then $z_5 \in E_8$ and $z_5^5 = 1$.

Now, we study the following subgroup $(E_8)^{z_5}$ of E_8 :

$$(E_8)^{z_5} = \{\alpha \in E_8 \mid z_5 \alpha = \alpha z_5\}.$$

Theorem 5.12.5. $(E_8)^{z_5} \cong (SU(5) \times SU(5))/Z_5, Z_5 = \{(E, E), (\zeta E, \zeta^2 E), (\zeta^2 E, \zeta^4 E), (\zeta^3 E, \zeta E), (\zeta^4 E, \zeta^3 E)\}, \zeta = \exp(2\pi i/5).$

Proof. We define mappings $\varphi_1, \varphi_2 : SU(5) \rightarrow E_8$ respectively by

$$\begin{aligned} \varphi_1(A)(C, D, \mathbf{x} \otimes \mathbf{a}, \mathbf{b} \otimes \mathbf{y}, \mathbf{c} \otimes \mathbf{z}, \mathbf{w} \otimes \mathbf{d}) \\ = (ACA^{-1}, D, (A\mathbf{x}) \otimes \mathbf{a}, (A\mathbf{b}) \otimes \mathbf{y}, ({}^t A^{-1} \mathbf{c}) \otimes \mathbf{z}, ({}^t A^{-1} \mathbf{w}) \otimes \mathbf{d}). \\ \varphi_2(B)(C, D, \mathbf{x} \otimes \mathbf{a}, \mathbf{b} \otimes \mathbf{y}, \mathbf{c} \otimes \mathbf{z}, \mathbf{w} \otimes \mathbf{d}) \\ = (C, BDB^{-1}, \mathbf{x} \otimes (B\mathbf{a}), \mathbf{b} \otimes ({}^t B^{-1} \mathbf{y}), \mathbf{c} \otimes (B\mathbf{z}), \mathbf{w} \otimes ({}^t B^{-1} \mathbf{d})). \end{aligned}$$

φ_1 and φ_2 are well-defined: $\varphi_1(A), \varphi_2(B) \in E_8$. Indeed, for $Z \in \mathfrak{su}(5)$, we have $(Z, 0) \in \mathfrak{g}_0$ and

$$\begin{aligned} \exp(\text{ad}(Z, 0))(C, D, \mathbf{x} \otimes \mathbf{a}, \mathbf{b} \otimes \mathbf{y}, \mathbf{c} \otimes \mathbf{z}, \mathbf{w} \otimes \mathbf{d}) \\ = (\exp(\text{ad}(Z))C, D, ((\exp Z)\mathbf{x}) \otimes \mathbf{a}, \\ ((\exp Z)\mathbf{b}) \otimes \mathbf{y}, ((\exp(-{}^t Z))\mathbf{c}) \otimes \mathbf{z}, ((\exp(-{}^t Z))\mathbf{w}) \otimes \mathbf{d}) \\ = (\text{Ad}(\exp Z)C, D, ((\exp Z)\mathbf{x}) \otimes \mathbf{a}, \\ ((\exp Z)\mathbf{b}) \otimes \mathbf{y}, ({}^t(\exp Z)^{-1} \mathbf{c}) \otimes \mathbf{z}, ({}^t(\exp Z)^{-1} \mathbf{w}) \otimes \mathbf{d}) \\ = \varphi_1(\exp Z)(C, D, \mathbf{x} \otimes \mathbf{a}, \mathbf{b} \otimes \mathbf{y}, \mathbf{c} \otimes \mathbf{z}, \mathbf{w} \otimes \mathbf{d}). \end{aligned}$$

Hence $\varphi_1(A) \in \text{Aut}(\mathfrak{e}_8^C) = E_8^C$. Using Lemma 5.11.1, we have

$$\langle \varphi(A)R_1, \varphi_1(A)R_2 \rangle = \langle R_1, R_2 \rangle.$$

Therefore $\varphi_1(A) \in E_8$. Similarly $\varphi_2(B) \in E_8$.

Now, we define a mapping $\varphi : SU(5) \times SU(5) \rightarrow E_8$ by

$$\varphi(A, B) = \varphi_1(A)\varphi_2(B).$$

Since $\varphi_1(A)$ and $\varphi_2(B)$ commute, φ is a homomorphism. We shall show that φ is onto. Since $(\mathfrak{e}_8)^{z_5} = \mathfrak{su}(5) \oplus \mathfrak{su}(5)$, the differential φ_* is onto. It is not difficult to see that

$$\text{Ker } \varphi = \{(E, E), (\zeta E, \zeta^2 E), (\zeta^2 E, \zeta^4 E), (\zeta^3 E, \zeta E), (\zeta^4 E, \zeta^3 E)\} = \mathbf{Z}_5.$$

Further, since $(E_8)^{z_5}$ is connected, φ is onto. Thus we have the isomorphism $(SU(5) \times SU(5))/\mathbf{Z}_5 \cong (E_8)^{z_5}$.

5.13. Non-compact exceptional Lie groups $E_{8(8)}$ and $E_{8(-24)}$ of type E_8

Let

$$\begin{aligned} \mathfrak{e}_{8(8)} &= \mathfrak{e}_{7(7)} \oplus \mathfrak{P}' \oplus \mathfrak{P}' \oplus \mathbf{R} \oplus \mathbf{R} \oplus \mathbf{R}, \quad (\text{where } \mathfrak{e}_{7(7)} = (\mathfrak{e}_7^C)^{\tau\gamma}), \\ \mathfrak{e}_{8(-24)} &= \mathfrak{e}_{7(-25)} \oplus \mathfrak{P} \oplus \mathfrak{P} \oplus \mathbf{R} \oplus \mathbf{R} \oplus \mathbf{R}, \quad (\text{where } \mathfrak{e}_{7(-25)} = (\mathfrak{e}_7^C)^\tau). \end{aligned}$$

For $R_1, R_2 \in \mathfrak{e}_{8(8)}$ or $\mathfrak{e}_{8(-24)}$, we define a Lie bracket $[R_1, R_2]$ as similar to \mathfrak{e}_8^C of Section 5.1. Now, we define groups $E_{8(8)}$ and $E_{8(-24)}$ by

$$\begin{aligned} E_{8(8)} &= \{\alpha \in \text{Iso}_R(\mathfrak{e}_{8(8)}) \mid \alpha[R_1, R_2] = [\alpha R_1, \alpha R_2]\}, \\ E_{8(-24)} &= \{\alpha \in \text{Iso}_R(\mathfrak{e}_{8(-24)}) \mid \alpha[R_1, R_2] = [\alpha R_1, \alpha R_2]\}. \end{aligned}$$

These groups can also be defined by

$$E_{8(8)} \cong (E_8^C)^{\tau\gamma}, \quad E_{8(-24)} \cong (E_8^C)^\tau.$$

Theorem 5.13.1. *The polar decompositions of the Lie groups $E_{8(8)}$ and $E_{8(-24)}$ are respectively given by*

$$\begin{aligned} E_{8(8)} &\simeq Ss(16) \times \mathbf{R}^{128}, \\ E_{8(-24)} &\simeq (SU(2) \times E_7)/\mathbf{Z}_2 \times \mathbf{R}^{112}. \end{aligned}$$

Proof. These are the facts corresponding to Theorems 5.8.7 and 5.7.6.

Theorem 5.13.2. *The centers of the groups $E_{8(8)}$ and $E_{8(-24)}$ are trivial.*

$$z(E_{8(8)}) = \{1\}, \quad z(E_{8(-24)}) = \{1\}.$$

References

- [1] K. Abe and I. Yokota, Realization of spaces $E_6/(U(1)Spin(10))$, $E_7/(U(1)E_6)$, $E_8/(U(1)E_7)$ and their volumes, Tokyo J. Math. 20(1997), 73-86.
- [2] K. Abe and I. Yokota, Volumes of compact symmetric spaces, Tokyo J. Math. 20(1997), 87-105.
- [3] M.F.Atiyah, R.Bott and A.Shapiro, Clifford modules, Topology 3, Suppl. 1(1964).
- [4] A. Borel and J. de Siebenthal, Les sous-groupes fermes de rang maximum des groupes de Lie clos, Comment. Math. 23(1949-50), 200-221.
- [5] C. Chevalley, Theory of Lie groups I, Princeton Univ. Press, 1946.
- [6] H. Freudenthal, Oktaven, Ausnahmegruppen und Oktavengeometrie, Math. Inst. Rijkuniv. te Utrecht, 1951.
- [7] H. Freudenthal, Beziehung der E_7 und E_8 zur Oktavenebene I, II, X, Nedel, Akad. Weten. Proc.Ser. A.57(1954), 218-230, 363-365, 66(1963), 457-471.
- [8] F. Gantmacher. On the classifications of real simple Lie groups, Rec. Math. J. 5(1939).
- [9] S. Gomyo, Realization of maximal subgroups of rank 8 of the simply connected compact simple Lie group of type E_8 , Tsukuba J. Math. 21(1997), 595-616.
- [10] S. Gomyo, Realizations of subgroups of type D_8 of connected exceptional simple Lie groups of type E_8 , Tsukuba J. Math. 23(1999), 585-615.
- [11] M. Goto and F. D. Grosshans, Semi-simple Lie algebras, Dekker, 1978.
- [12] T. Imai, On bounded symmetric domains of exceptional type, J. Fac. Sci. Shinshu Univ. 16(1981), 65-96.
- [13] T. Imai and I. Yokota, Non-compact simple Lie group $E_{8(-24)}$ of type E_8 , J. Fac. Sci. Shinshu Univ. 15(1980), 53-76.
- [14]* † T. Imai and I. Yokota, Another definitions of exceptional simple Lie groups of type $E_{7(-25)}$ and $E_{7(-133)}$, J. Fac. Sci. Shinshu Univ. 15(1980), 47-52.
- [15]* † T. Imai and I. Yokota, Simply connected compact simple Lie group $E_{7(-133)}$ of type E_7 , J. Math. Kyoto Univ. 21(1981), 383-395.
- [16] T. Imai and I. Yokota, Simply connected compact simple Lie group $E_{8(-248)}$ of type E_8 , J. Math. Kyoto Univ. 21(1981), 741-762.

- [17] S. Kaneyuki, On the subalgebra \mathfrak{g}_0 and \mathfrak{g}_{ev} of semisimple Lie algebra, J. Math. Soc. Japan, 45(1993), 1-19.
- [18] Y. Matsushima, Some remarks on exceptional Lie group F_4 , Nagoya Math. J. 4(1954), 83-88.
- [19] Y. Matsushima, Theory of Lie algebras (in Japanese), Kyoritsu-shuppan, Tokyo, 1957.
- [20]* K. Mitsuishi and I. Yokota, Volume formula of compact simple Lie groups and values of τ , J. Fac. Sci. Shinshu Univ. 29(1994), 69-70.
- [21] T. Miyasaka, Adjoint orbit types of compact exceptional Lie group G_2 in their Lie algebra, Math. J. Okayama Univ. 43(2001), 17-180.
- [22] T. Miyasaka, O. Schukuzawa and I. Yokota, Spinor-generators of compact exceptional Lie groups F_4, E_6 and E_7 , Tsukuba J. Math. 22(1998), 705-721.
- [23] T. Miyasaka, O. Yasukura and I. Yokota, Diagonalization of an element P of \mathfrak{P}^C by the compact Lie group E_7 , Tsukuba J. Math. 22(1998), 687-703.
- [24] T. Miyasaka and I. Yokota, Orbit types of the compact Lie group E_7 in the Freudenthal vector space \mathfrak{P}^C , Tsukuba J. Math. 23(1999), 229-234.
- [25] T. Miyashita, Decomposition of spinor groups by the involution σ' in exceptional Lie groups, Math. J. Okayama Univ, 44(2001), 151-158.
- [26] T. Miyashita, Fixed points subgroups by two involutive automorphisms σ, γ of compact exceptional Lie groups F_4, E_6 and E_7 , Tsukuba J. Math. 27(2003), 199-215.
- [27] T. Miyashita and I. Yokota, 2-graded decompositions of exceptional Lie algebra \mathfrak{g} and group realizations of $\mathfrak{g}_{ev}, \mathfrak{g}_0$, Part III, $G = E_8$, Japanese J. Math. 26-1(2000), 31-50.
- [28] T. Miyashita and I. Yokota, Fixed points subgroups $G^{\sigma, \sigma'}$ by two involutive automorphisms σ, σ' of compact exceptional Lie groups F_4, E_6 and E_7 , Tsukuba J. Math. 24(2001), 135-149.
- [29] T. Miyashita and I. Yokota, An explicit isomorphism between $(\mathfrak{e}_8)^{\sigma, \sigma'}$ and $\mathfrak{so}(8) \oplus \mathfrak{so}(8)$ as Lie algebras, Math. J. Toyama Univ. 24(2001), 151-158.
- [30] T. Miyashita and I. Yokota, 3-graded decompositions of exceptional Lie algebra \mathfrak{g} and group realizations of $\mathfrak{g}_{ev}, \mathfrak{g}_0$ and \mathfrak{g}_{ed} , Part II, $G = E_7$, Case 1, J. Kyoto Univ. 46-2(2006), 383-413, Case 2, 3, 46-4(2006), 805-832, Case 5, 47-1(2007), 121-128.

- [31] T. Miyashita and I. Yokota, 3-graded decompositions of exceptional Lie algebra \mathfrak{g} and group realizations of \mathfrak{g}_{ev} , \mathfrak{g}_0 and \mathfrak{g}_{ed} , Part III, $G = E_8$, J. Kyoto Univ. to appear.
- [32] P. K. Rasevskii, A theorem on the connectedness of a subgroup of a simply connected Lie group commuting with any its automorphis A theorem on the connectms, Trans. Moscow Math. Soc. 30(1974), 3-22.
- [33] J. Sekiguchi, Fundamental group of semisimple symmetric spaces, Advanced Studies to pure Math. 14(1988), 519-529.
- [34] O. Shukuzawa, Embeddings of unit spheres into exceptional groups of type G_2 and homeomorphisms $(G_2)^C/SL(3, C) \simeq (S^C)^6, G_{2(2)}/SL(3, \mathbf{R}) \simeq S_{3,4}$, Math. J. Toyama Univ. 24(2001), 159-171.
- [35] O. Shukuzawa, Diagonalization of elements of Freudenthal \mathbf{R} -vector space and split Freudenthal \mathbf{R} -space, Tsukuba J. Math. 27(2003), 113-128.
- [36] O. Shukuzawa, Orbit space of split Freudenthal vector space, Yokohama Math. J. 50(2003), 1-10.
- [37] O. Shukuzawa and I. Yokota, Non-compact simple Lie group $E_{6(6)}$ of type E_6 , J. Fac. Sci. Shinshu Univ. 14(1979), 1-13.
- [38] O. Shukuzawa and I. Yokota, Non-compact simple Lie group $E_{6(-14)}$ and $E_{6(2)}$ of type E_6 , J. Fac. Sci. Shinshu Univ. 14(1979), 15-28.
- [39] I. Yokota, Embedding of projective spaces into elliptic projective Lie groups, Proc. Japan Acad. 35(1959), 281-283.
- [40] I. Yokota, A note on $Spin(9)$, J. Fac. Sci. Shinshu Univ. 3(1968), 61-70.
- [41] I. Yokota, Exceptional Lie group F_4 and its representation rings, J. Fac. Sci. Shinshu Univ. 3(1968), 35-60.
- [42] I. Yokota, On a non-compact simple Lie group $F_{4,1}$ of type F_4 , J. Fac. Sci. Shinshu Univ. 10(1975), 71-80.
- [43] I. Yokota, Non-compact simple Lie group G_2' of type G_2 , J. Fac. Sci. Shinshu Univ. 12(1977), 45-52.
- [44] I. Yokota, Non-compact simple Lie group $F_{4,2}$ of type F_4 , J. Fac. Sci. Shinshu Univ. 12(1977), 53-64.
- [45] I. Yokota, Subgroups of type A_2 -series in the exceptional Lie group G_2 , J. Fac. Sci. Shinshu Univ. 14(1979), 87-94.

- [46][†] I. Yokota, Simply connected compact simple Lie group $E_{6(-78)}$ of type E_6 and its involutive automorphisms, J. Math. Kyoto Univ. 20(1980), 447-473.
- [47]^{*} I. Yokota, Subgroup $(SU(8))/\mathbf{Z}_2$ of compact simple Lie group E_7 and non-compact simple Lie group $E_{7(7)}$ of type E_7 , Math. J. Okayama Univ. 24(1982), 53-71.
- [48] I. Yokota, Realization of automorphisms σ of order 3 and G^σ of compact exceptional Lie groups G , I, $G = G_2, F_4, E_6$, J. Fac. Sci. Shinshu Univ. 20(1985), 131-144.
- [49] I. Yokota, Realization of involutive automorphisms σ of exceptional Lie groups G , Part I, $G = G_2, F_4$ and E_6 , Part II, $G = E_7$, Part III, $G = E_8$, Tsukuba J. Math. 14(1990), 185-223, 14(1990), 379-404, 15(1991), 301-314.
- [50] I. Yokota, Orbit types of the compact Lie group E_6 in the complex exceptional Jordan algebra \mathfrak{J}^C , Proc. Symp. on non-associative algebras, Japan, 1991, 353-359.
- [51] I. Yokota, 2-graded decompositions of exceptional Lie algebra \mathfrak{g} and group realizations of $\mathfrak{g}_{\text{ev}}, \mathfrak{g}_0$, Part I, $G = G_2, F_4, E_6$, Part II, $G = E_7$, Japan. J. Math. 24(1998), 258-296, 25(1999), 155-179.
- [52] I. Yokota, 3-graded decompositions of exceptional Lie algebra \mathfrak{g} and group realizations of $\mathfrak{g}_{\text{ev}}, \mathfrak{g}_0$ and \mathfrak{g}_{ed} , Part I, $G = G_2, F_4, E_6$, J. Math. Kyoto Univ. 41-3(2001), 449-494.
- [53] I. Yokota, T. Imai and O. Yasukura, On the homogeneous space E_8/E_7 , J. Math., Kyoto Univ. 23(1983), 467-473.
- [54] I. Yokota, T. Ishihara and O. Yasukura, Subgroup $((SU(3) \times SU(6))/Z_2).Z_2$ of the simply connected compact simple Lie group E_7 , J. Math. Kyoto Univ. 23(1983), 715-737.
- [55] I. Yokota and O. Shukuzawa, Automorphisms of order 4 of the simply connected compact Lie group E_6 , Tsukuba J. Math. 15(1991), 451-463.
- [56] I. Yokota and O. Yasukura, Subgroup $(SU(2) \times Spin(12))/\mathbf{Z}_2$ of compact simple Lie group E_7 and non-compact simple group $E_{7,\sigma}$ of type $E_{7(-5)}$, Hiroshima Univ. J. 12(1982), 59-76.
- [57] I. Yokota and O. Yasukura, Non-compact simple Lie group $E_{8(8)}$, Tsukuba J. Math. 10(1986), 331-349.
- [58] I. Yokota, Groups and Topology (in Japanese), Shokabo, Yokyo, 1971.

- [59] I. Yokota, Classical linear simple Lie groups (in Japanese), Gendai-sugakushya, Kyoto, 1990.
- [60] I. Yokota, Exceptional linear simple Lie groups (in Japanese), Gendai-sugakushya, Kyoto, 1992.

* has some errors

† --- and τ are written by the same notation --- .